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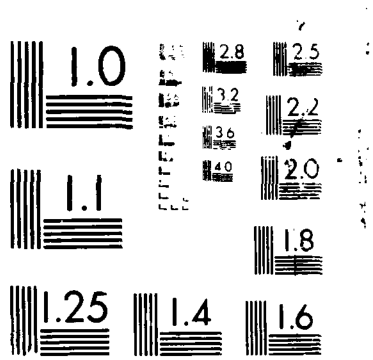
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A CONTINUOUS ANALOGUE OF STURM SEQUENCES IN THE CONTEXT OF STURM-LIOUVILLE EQUATIONS

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A shooting method is presented for finding the n^{th} eigenvalue and eigenfunction of a Sturm-Lioville equation, in which the eigenvalue occurs nonlinearly. The method is verified in two ways: by applying the Sturm comparision and oscillation theorems to the continuous problem; and by applying Sturm sequences to a discretization. The method works for general (separated) boundary conditions, and provides an a-posteriori error estimate for the approximate eigenvalue. Analogues of the Sturm comparison, oscillation and separation theorems are proved for the discrete problem. A related method, which involves critical lengths in (over)

20. the invariant imbedding method, is shown to be incorrect for general boundary conditions.

A Continuous Analogue of Sturm Sequences
in the Context of Sturm-Liouville Equations

by

L. Greenberg[1] and I. Babuška[2]

Abstract. A shooting method is presented for finding the n^{th} eigenvalue and eigenfunction of a Sturm-Liouville equation, in which the eigenvalue occurs nonlinearly. The method is verified in two ways: by applying the Sturm comparison and oscillation theorems to the continuous problem; and by applying Sturm sequences to a discretization. The method works for general (separated) boundary conditions, and provides an a-posteriori error estimate for the approximate eigenvalue. Analogues of the Sturm comparison, oscillation and separation theorems are proved for the discrete problem. A related method, which involves critical lengths in the invariant imbedding method, is shown to be incorrect for general boundary conditions.

Key words: Eigenvalue, eigenfunction, Sturm-Liouville equation, shooting method, comparison theorems, oscillation theorem, discretization, Sturm sequence.

AMS (MOS) classification: 65L15

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1. Sturm-Liouville Equations.

Consider the Sturm-Liouville eigenvalue problem

$$(1.1)\qquad \begin{cases} (p(x,\lambda)u')' + q(x,\lambda)u = 0, & \text{for } 0 \le x \le 1, \\ \alpha_0(\lambda)u(0) + \beta_0(\lambda)u'(0) = 0, \\ \alpha_1(\lambda)u(1) + \beta_1(\lambda)u'(1) = 0. \end{cases}$$

The main objective of this paper is to present a shooting method for the eigenvalues and eigenfunctions of (1.1), and to prove its validity. The method is based on oscillation, and is related to the critical (or characteristic) lengths in the invariant imbedding method. However, a simple counting of critical lengths does not produce a correct algorithm (for general, separated boundary conditions). We shall discuss this in detail in §6. Our method can aim for the n^{th} eigenvalue without consideration of other eigenvalues. It provides an a-posteriori error estimate for the approximate eigenvalue. The method is a generalization of that used by Porter and Reiss [13], [14], for the problem

$$(1.2)\qquad \begin{cases} \left[\dfrac{u'}{(\omega-\lambda u^{o}(x))^2}\right]' + \left[\dfrac{1}{c(x)^2} - \dfrac{\lambda^2}{(\omega-\lambda u^{o}(x))^2}\right]u = 0, & \text{for } 0 \le x \le d, \\ u(0) = 0 = u'(d), \end{cases}$$

which arises in acoustics. (Here, $u^{o}(x)$ is a given function.)

The shooting method will be described in §2, and its validity will be proved using the Sturm comparison and oscillation theorems. The problem will be discretized in §3, and another verification of the shooting method will be given in §4, by applying Sturm sequences to the discrete problem. This process of "taking the limit" of a numerical method (applied to a discretization) is sometimes referred to as finding a "closure of an algorithm" (see

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Babuška [1]). Such closures give insight into the numerical method, and often lead to more flexible and adaptable procedures than the finite difference or finite element method. An example of this is the double sweep method (see, for example, Babuška, Präger and Vitásek [4], Babuska and Majer [2], Keller and Lentini [12]). In §5, analogues of the Sturm comparison, oscillation and separation theorems will be given for the discrete problem. In §6, we provide a cautionary note concerning a related shooting method which can fail, although its appearance of validity is seductive.

We shall need to make certain assumptions about the coefficient functions in (1.1). As indicated below, these occur in three categories. The standard assumptions will always be implicitly assumed without mention. The monotonicity and limit assumptions occur in the Sturm comparison and oscillation theorems. They will be explicitly assumed when needed. In the following, λ will vary in an interval (Λ_1, Λ_2). We do not exclude the possibilities $\Lambda_1 = -\infty$, $\Lambda_2 = \infty$.

Standard Assumptions

(S1) $p(x,\lambda)$, $\frac{\partial}{\partial x}p(x,\lambda)$ and $q(x,\lambda)$ are continuous functions on $[0,1]\times(\Lambda_1,\Lambda_2)$.

(S2) $p(x,\lambda) \ge k > 0$, for $0 \le x \le 1$, $\Lambda_1 < \lambda < \Lambda_2$.

(S3) $\alpha_0(\lambda)$, $\beta_0(\lambda)$, $\alpha_1(\lambda)$, $\beta_1(\lambda)$ are continuous on (Λ_1,Λ_2).

(S4) $\alpha_i(\lambda)^2 + \beta_i(\lambda)^2 \neq 0$, for $i = 0,1$ and $\Lambda_1 < \lambda < \Lambda_2$.

(S5) For $i = 0,1$, either $\beta_i(\lambda) \equiv 0$ or $\beta_i(\lambda) > 0$, for $\Lambda_1 < \lambda < \Lambda_2$.(*)

Monotonicity Assumptions

(M1) For each x, $q(x,\lambda)$ is a strictly increasing function of λ.

(M2) For each x, $p(x,\lambda)$ is a nonincreasing function of λ.

(*) If $\lambda_+ = \Lambda_1$ in (L3), then we assume that (S5) is valid for $\Lambda_1 \le \lambda < \Lambda_2$.

(M3) If $\beta_0(\lambda) \neq 0$, then $p(0,\lambda)\frac{\alpha_0(\lambda)}{\beta_0(\lambda)}$ is a nondecreasing function.

(M4) If $\beta_1(\lambda) \neq 0$, then $p(0,\lambda)\frac{\alpha_1(\lambda)}{\beta_1(\lambda)}$ is a nonincreasing function.

We shall use the following notation:

$$(1.3)\qquad \begin{cases} p^*(\lambda) = \max\limits_{0\le x\le 1} p(x,\lambda), & p_*(\lambda) = \min\limits_{0\le x\le 1} p(x,\lambda), \\ q^*(\lambda) = \max\limits_{0\le x\le 1} q(x,\lambda), & q_*(\lambda) = \min\limits_{0\le x\le 1} q(x,\lambda). \end{cases}$$

Note that $p^*(\lambda) \ge p_*(\lambda) \ge k > 0$, by assumption (S2).

<u>Limit Assumptions</u>

(L1) $\lim\limits_{\lambda\to\Lambda_2} \dfrac{q_*(\lambda)}{p^*(\lambda)} = \infty.$

(L2) $\lim\limits_{\lambda\to\Lambda_1} \dfrac{q^*(\lambda)}{p_*(\lambda)} = -\infty.$

(L3) There is a number λ_+ in (Λ_1,Λ_2), so that $\alpha_0(\lambda_+)\beta_0(\lambda_+) \le 0$, $\alpha_1(\lambda_+)\beta_1(\lambda_+) \ge 0$, and $q^*(\lambda_+) \le 0$. (If the coefficient functions in (1.1) can be extended continuously to $\lambda = \Lambda_1$, we may take $\lambda_+ = \Lambda_1$.)[(*)]

Note that assumption (L3) implies that for $\lambda = \lambda_+$, the operator in (1.1) is negative semidefinite.

<u>Remark</u>. In the acoustics problem (1.2), $q(x,\lambda)$ is a decreasing function of λ, and $p(x,\lambda)$ is an increasing function of λ, if $\omega > 0$, $u^o(x) > 0$ and $\omega - \lambda u^o(x) > 0$. However, the shooting method is valid if the words "increasing" and "decreasing" are interchanged in assumptions (M1) - (M4), while Λ_1 and Λ_2 are interchanged in assumptions (L1) - (L3).

We shall conclude this section by recalling the Sturm comparison and oscillation theorems. These theorems (with somewhat different notation) can be found in Bôcher [5, Chap. 3] and Ince [10, Chap. 10]. The comparison

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[(*)] If $\lambda_+ = \Lambda_1$ in (L3), then we assume that (S5) is valid for $\Lambda_1 \le \lambda < \Lambda_2$.

theorems deal with the initial value problem obtained from (1.1) by omitting the boundary condition at $x = 1$. This can be formulated as follows:

$$(1.4) \qquad \begin{cases} (p(x,\lambda),u')' + q(x,\lambda)u = 0, & \text{for } 0 \le x \le 1, \\ u(0) = \beta_0(\lambda), \ u'(0) = -\alpha_0(\lambda). \end{cases}$$

Let $u(x,\lambda)$ denote the solution of (1.4). Note that (for fixed λ) the zeros of $u(x,\lambda)$ are simple, since (1.4) is a second order differential equation.

Theorem A (First Comparison Theorem). Suppose that (1.4) satisfies the monotonicity assumptions (M1), (M2), and (M3). Then, for $\lambda_1 < \lambda_2$, $u(x,\lambda_2)$ has at least as many zeros as $u(x,\lambda_1)$ in the interval $0 < x \le 1$, and the i^{th} zero of $u(x,\lambda_2)$ is less than the i^{th} zero of $u(x,\lambda_1)$.

Theorem B (Second Comparison Theorem). Suppose that (1.4) satisfies the monotonicity assumptions (M1), (M2), and (M3). Let $0 < x_0 \le 1$, and suppose that $u(x_0,\lambda) \ne 0$ for $\lambda_1 < \lambda < \lambda_2$. Then $p(x_0,\lambda)\frac{u'(x_0,\lambda)}{u(x_0,\lambda)}$ is a strictly decreasing function of λ in the interval $\lambda_1 < \lambda < \lambda_2$.

Remark 1. The comparison theorems are usually stated in terms of two differential equations:

$$(1.5) \qquad \begin{cases} (p_0(x)u_0')' + q_0(x)u_0 = 0, & \text{for } 0 \le x \le 1, \\ u_0(0) = \beta_0, \ u_0'(0) = -\alpha_0, \end{cases}$$

and

$$(1.6) \qquad \begin{cases} (p_1(x)u_1')' + q_1(x)u_1 = 0, & \text{for } 0 \le x \le 1, \\ u_1(0) = \beta_1, \ u_1'(0) = -\alpha_1, \end{cases}$$

where it assumed that

(1.7) $\quad p_0(x) \geq p_1(x), \quad q_0(x) < q_1(x) \quad \text{and} \quad p_0(0)\frac{\alpha_0}{\beta_0} \leq p_1(0)\frac{\alpha_1}{\beta_1}.$

(For the second comparison theorem, we also assume that $u_0(x)$ and $u_1(x)$ have the same number of zeros in the interval $(0, x_0)$, and $u_0(x_0) \neq 0$, $u_1(x_0) \neq 0$.) However, the equations (1.5) and (1.6) can be embedded in a continuous family of equations:

$$(1.8) \qquad \begin{cases} (p(x,\lambda)u')' + q(x,\lambda)u = 0, & \text{for } 0 \leq x \leq 1, \\ u(0) = \beta(\lambda), \quad u'(0) = -\alpha(\lambda), & \end{cases}$$

which satisfies the monotonicity assumptions (M1), (M2), and (M3) for $0 \leq \lambda \leq 1$, and such that $p(x,0) = p_0(x)$, $q(x,0) = q_0(x)$, $p(x,1) = p_1(x)$, $q(x,1) = q_1(x)$, $\frac{\alpha(0)}{\beta(0)} = \frac{\alpha_0}{\beta_0}$ and $\frac{\alpha(1)}{\beta(1)} = \frac{\alpha_1}{\beta_1}$. We can take $p(x,\lambda) = (1-\lambda)p_0(x) + \lambda p_1(x)$, $q(x,\lambda) = (1-\lambda)q_0(x) + \lambda q_1(x)$, and choose $\alpha(\lambda)$, $\beta(\lambda)$ so that

$$p(0,\lambda)\frac{\alpha(\lambda)}{\beta(\lambda)} = (1-\lambda)p_0(0)\frac{\alpha_0}{\beta_0} + \lambda p_1(0)\frac{\alpha_1}{\beta_1}.$$

Namely, take $\alpha(\lambda) = (1-\lambda)p_0(0)\alpha_0\beta_1 + \lambda p_1(0)\alpha_1\beta_0$, and $\beta(\lambda) = \beta_0\beta_1 p(0,\lambda)$. It then follows that the solution $u(x,\lambda)$ of (1.8) satisfies

$$(1.9) \qquad u(x,0) = c_0 u_0(x), \quad u(x,1) = c_1 u_1(x),$$

where $c_0 = p_0(0)\beta_1$, $c_1 = p_1(0)\beta_0$. Thus $u(x,0)$ and $u_0(x)$ have the same number of zeros in $(0,1)$, as do $u(x,1)$ and $u_1(x)$. Also $p(x,0)\frac{u'(x,0)}{u(x,1)} = p_1(x)\frac{u_1'(x)}{u_1(x)}$, for $0 \leq x \leq 1$. We shall feel free to use either form of the comparison theorems.

<u>Remark 2</u>. If the strict inequality $q_0(x) < q_1(x)$ is changed to $q_0(x) \leq q_1(x)$ in (1.7), then the conclusions are changed analogously: the first comparison theorem concludes that $u_1(x)$ has at least as many zeros as

$u_0(x)$ in the interval $0 < x \le 1$, and the i^{th} zero of $u_1(x)$ is less than or equal to the i^{th} zero of $u_0(x)$. The second comparison theorem concludes that $p_1(x)\frac{u_0'(x_0)}{u_0(x_0)} \ge p_1(x_0)\frac{u_1'(x_0)}{u_1(x_0)}$.

Theorem C (Oscillation Theorem). Suppose that (1.1) satisfies the monotonicity assumptions (M1) - (M4) and the limit assumption (L1). Then the eigenvalues of (1.1) form an infinite, increasing sequence $\lambda_m < \lambda_{m+1} < \lambda_{m+1} < \dots$, which tends to Λ_2. The eigenfunction $\varphi_k(x)$, corresponding to λ_k, has exactly $k-1$ zeros in the interval $(0,1)$. Furthermore, suppose that (1.1) satisfies either the limit assumption (L2) or (L3). Then the sequence of eigenvalues begins with λ_1, whose eigenfunction $\varphi_1(x)$ has no zeros in $(0,1)$.

2. A Shooting Method for Eigenvalues and Eigenfunctions.

An eigenvalue of (1.1) will be denoted λ_k, and called the k^{th} eigenvalue, if its corresponding eigenfunction $\varphi_k(x)$ has exactly $k-1$ zeros in $(0,1)$. This notation has already been used in the Sturm oscillation theorem, which gives sufficient conditions for λ_k to exist for large k, or for all $k \geq 1$.

Definition 2.1. For a given λ in (Λ_1, Λ_2), let $u(x,\lambda)$ denote the solution of the initial value problem (1.4). Furthermore, let

(1) $N_o(\lambda)$ be the number of zeros of $u(x,\lambda)$ in $(0,1)$,

(2)
$$\bar{u}(\lambda) = \alpha_1(\lambda)u(1,\lambda) + \beta_1(\lambda)u'(1,\lambda),$$

(3)
$$\sigma(\lambda) = \begin{cases} 0 & \text{if } u(1,\lambda)\bar{u}(\lambda) > 0 \text{ or } \bar{u}(\lambda) = 0, \\ 1 & \text{if } u(1,\lambda)\bar{u}(\lambda) \leq 0 \text{ and } \bar{u}(\lambda) \neq 0, \end{cases}$$

(4)
$$N(\lambda) = N_o(\lambda) + \sigma(\lambda).$$

Remark. In the above formulation, $N(\lambda)$ is the number of zeros of $u(x,\lambda)$ in $(0,1)$, with a correction which depends on the boundary condition at $x = 1$. If $\beta_1(\lambda) \neq 0$, an equivalent formulation is: $N(\lambda) = M_o(\lambda) + \rho(\lambda)$, where $M_o(\lambda)$ is the number of zeros of $u(x,\lambda)$ in $(0,1]$, and

$$\rho(\lambda) = \begin{cases} 0 & \text{if } u(1,\lambda)\bar{u}(\lambda) \geq 0 \\ 1 & \text{if } u(1,\lambda)\bar{u}(\lambda) < 0. \end{cases}$$

The shooting method will use the oscillation counter $N(\lambda)$ in an essential way. The following theorem gives the main properties of $N(\lambda)$.

Theorem 2.1 (Shooting Theorem). Suppose that (1.1) satisfies the monotonicity assumptions (M1) - (M4). Let $\lambda' < \lambda''$ be numbers in $(\Lambda_1, \Lambda_2)^{(*)}$. Then:

(1) The interval $[\lambda', \lambda'')$ contains exactly $N(\lambda'') - N(\lambda')$ eigenvalues of (1.1).

(2) If $N(\lambda') = j < k = N(\lambda'')$, then the eigenvalues $\lambda_{j+1}, \lambda_{j+2}, \ldots, \lambda_k$ exist, and $\lambda' \le \lambda_{j+1} < \lambda_{j+2} < \ldots < \lambda_k < \lambda''$.

If (1.1) also satisfies either the limit assumption (L2) or (L3) then

(3) For $\Lambda_1 < \lambda < \Lambda_2$, (1.1) has exactly $N(\lambda)$ eigenvalues in the interval $[\Lambda_1, \lambda)^{(**)}$.

The proof of this theorem is closely intertwined with the proof of the Sturm oscillation theorem. We shall begin with a short discussion and two lemmas, which will lead to the proof of the theorem.

Let $u(x,\lambda)$ be the solution of the initial value problem (1.4). Sturm's first comparison theorem states that, as λ increases, $u(x,\lambda)$ does not lose zeros. It may acquire new zeros, which first appear at the endpoint $x = 1$, and move toward $x = 0$. Suppose that the k^{th} zero appears when $\lambda = \mu_k$. The μ_k form an increasing sequence $\mu_m < \mu_{m+1} < \ldots$ (possibly a finite, or even empty sequence). The solution $u(x,\lambda)$ has exactly $k-1$ zeros in $(0,1)$, for $\mu_{k-1} < \lambda \le \mu_k$ (assuming that μ_{k-1} and μ_k exist). The second comparison theorem implies that $p(1,\lambda)\frac{u'(1,\lambda)}{u(1,\lambda)}$ is a strictly decreasing function for $\mu_{k-1} < \lambda < \mu_k$. Clearly, this function decreases from ∞ to $-\infty$.

(*) If the coefficient functions in (1.1) can be extended continuously to $\lambda = \Lambda_1$, then λ' may be taken in $[\Lambda_1, \Lambda_2)$.

(**) The interval $[\Lambda_1, \lambda)$ in conclusion (3) may be replaced by (Λ_1, λ) unless $\lambda_+ = \Lambda_1 = \lambda_1$ in asssumption (L3). See the remark after Lemma 2.2, below.

On the other hand, the monotonicity assumption (M4) states that $p(1,\lambda)\frac{\alpha_1(\lambda)}{\beta_1(\lambda)}$ is a nonincreasing function (assuming that $\beta_1(\lambda) \not\equiv 0$), so that $-p(1,\lambda)\frac{\alpha_1(\lambda)}{\beta_1(\lambda)}$ is a nondecreasing function. Therefore there is a unique number λ_k, $\mu_{k-1} < \lambda_k < \mu_k$, such that for $\lambda = \lambda_k$, $p(1,\lambda)\frac{u'(1,\lambda)}{u(1,\lambda)} = -p(1,\lambda)\frac{\alpha_1(\lambda)}{\beta_1(\lambda)}$, or $\alpha_1(\lambda)u(1,\lambda) + \beta_1(\lambda)u'(1,\lambda) = 0$. Thus $\lambda = \lambda_k$ is the k^{th} eigenvalue of (1.1).

Note that if $\beta_1(\lambda) \equiv 0$, then $\lambda_k = \mu_k$ is the k^{th} eigenvalue. In this case $\sigma(\lambda)$ is always zero, and $N(\lambda) = N_o(\lambda)$. Thus $N(\lambda) = k - 1$, for $\mu_{k-1} < \lambda \le \mu_k$.

<u>Lemma 2.1</u>. Suppose that (1.1) satisfies the monotonicity assumptions (M1) - (M4), and $\beta_1(\lambda) \not\equiv 0$. Then $N(\lambda) = k - 1$ for $\mu_{k-1} < \lambda \le \lambda_k$, and $N(\lambda) = k$ for $\lambda_k < \lambda \le \mu_k$.

<u>Proof</u>. Because $p(1,\lambda)\frac{u'(1,\lambda)}{u(1,\lambda)}$ is a strictly decreasing function in the interval $\mu_{k-1} < \lambda < \mu_k$, and $-p(1,\lambda)\frac{\alpha_1(\lambda)}{\beta_1(\lambda)}$ is a nondecreasing function, we see that $p(1,\lambda)\frac{u'(1,\lambda)}{u(1,\lambda)} > -p(1,\lambda)\frac{\alpha_1(\lambda)}{\beta_1(\lambda)}$ for $\mu_{k-1} < \lambda < \lambda_k$, and $p(1,\lambda)\frac{u'(1,\lambda)}{u(1,\lambda)} < -p(1,\lambda)\frac{\alpha_1(\lambda)}{\beta_1(\lambda)}$ for $\lambda_k < \lambda < \mu_k$. This implies that $\beta_1(\lambda)u(1,\lambda)[\alpha_1(\lambda)u(1,\lambda) + \beta_1(\lambda)u'(1,\lambda)] > 0$ for $\mu_{k-1} < \lambda < \lambda_k$, and $\beta_1(\lambda)u(1,\lambda)[\alpha_1(\lambda)u(1,\lambda) + \beta_1(\lambda)u'(1,\lambda)] < 0$ for $\lambda_k < \lambda < \mu_k$. Recalling that $\beta_1(\lambda) > 0$ and $\bar{u}(\lambda) = \alpha_1(\lambda)u(1,\lambda) + \beta_1(\lambda)u'(1,\lambda)$, we see that $u(1,\lambda)\bar{u}(\lambda) > 0$ for $\mu_{k-1} < \lambda < \lambda_k$, and $u(1,\lambda)\bar{u}(\lambda) < 0$ for $\lambda_k < \lambda < \mu_k$. Referring to Definition 2.1, we see that $\sigma(\lambda) = 0$ for $\mu_{k-1} < \lambda \le \lambda_k$, and $\sigma(\lambda) = 1$ for $\lambda_k < \lambda \le \mu_k$. Since $N_o(\lambda) = k - 1$ for $\mu_{k-1} < \lambda \le \mu_k$ and $N(\lambda) = N_o(\lambda) + \sigma(\lambda)$, it follows that $N(\lambda) = k - 1$ for $\mu_{k-1} < \lambda \le \lambda_k$, and $N(\lambda) = k$ for $\lambda_k < \lambda \le \mu_k$. Q.E.D.

Lemma 2.2.

(1) If (1.1) satisfies the monotonicity assumptions (M3), (M4) and the limit assumption (L2), then $N(\lambda) = 0$, for λ near Λ_1.

(2) If (1.1) satisfies the limit assumption (L3), then $N(\lambda_+) = 0$.

Proof. (1) We shall use the first and second comparison theorems to compare the equations

$$\text{(2.1)} \qquad \begin{cases} (p(x,\lambda),u')' + q(x,\lambda)u = 0, & \text{for } 0 \le x \le 1, \\ u(0) = \beta_0(\lambda),\ u'(0) = -\alpha_0(\lambda). \end{cases}$$

and

$$\text{(2.2)} \qquad \begin{cases} (p_*(\lambda)v')' + q^*(\lambda)v = 0, & \text{for } 0 \le x \le 1, \\ v(0) = \beta(\lambda),\ v'(0) = -\alpha, \end{cases}$$

where

$$\text{(2.3)} \qquad \alpha = p(0,\lambda_0)\alpha_0(\lambda_0),\ \beta(\lambda) = p_*(\lambda)\beta_0(\lambda_0).$$

Here, λ_0 is a fixed number in (Λ_1,Λ_2), and $\Lambda_1 < \lambda < \lambda_0$. (See (1.3) for the notation $p_*(\lambda)$ and $q^*(\lambda)$.) Note that $p(x,\lambda) \ge p_*(\lambda)$, $q(x,\lambda) \le q^*(\lambda)$, and by monotonicity assumption (M3),

$$p(0,\lambda)\frac{\alpha_0(\lambda)}{\beta_0(\lambda)} \le p(0,\lambda_0)\frac{\alpha_0(\lambda_0)}{\beta_0(\lambda_0)} = p_*(\lambda)\frac{p(0,\lambda_0)\alpha_0(\lambda_0)}{p_*(\lambda)\beta_0(\lambda_0)} = p_*(\lambda)\frac{\alpha}{\beta(\lambda)}.$$

Therefore the comparison theorems apply (as in Remark 2 after Theorem B in §1). Thus, $u(x)$ does not have more zeros than $v(x)$ in $(0,1]$, and $p(1,\lambda)\frac{u'(1)}{u(1)} \ge p_*(\lambda)\frac{v'(1)}{v(1)}$, if $u(x)$ and $v(x)$ have the same number of zeros in $(0,1)$ and $u(1) \ne 0$, $v(1) \ne 0$. (We are sometimes suppressing the λ, and denoting $u(x,\lambda) = u(x)$, $v(x,\lambda) = v(x)$.)

The limit assumption (L2) states that $\lim_{\lambda\to\Lambda_1} \frac{q^*(\lambda)}{p_*(\lambda)} = -\infty$. Therefore, for

λ near Λ_1, $\frac{q^*(\lambda)}{p_*(\lambda)} = -s^2$, where $s = s(\lambda) = \sqrt{-\frac{q^*(\lambda)}{p_*(\lambda)}}$, and $\lim_{\lambda\to\Lambda_1} s(\lambda) = \infty$.

Equation (2.2) is equivalent to

$$\begin{cases} v'' - s^2 v = 0 \\ v(0) = \beta, \ v'(0) = -\alpha, \end{cases} \tag{2.4}$$

whose solution is

$$v = \beta \cosh sx - \frac{\alpha}{s} \sinh sx. \tag{2.5}$$

Recall that either $\beta_0(\lambda) \equiv 0$ or $\beta_0(\lambda) > 0$. If $\beta_0(\lambda) \equiv 0$, then $\beta(\lambda) = p_*(\lambda)\beta_0(\lambda_0) = 0$, and $\alpha = p(0,\lambda_0)\alpha(\lambda_0) \neq 0$. In this case $v = -\frac{\alpha}{s}\sinh sx$ has no zeros in $(0,1]$, so u doesn't have any zeros there either. If $\beta_0(\lambda) > 0$, then $\beta > 0$. In fact, β is bounded away from 0, because the standard assumption (S2) implies that $p_*(\lambda) \geq k > 0$, so $\beta(\lambda) = p_*(\lambda)\beta_0(\lambda_0) \geq k\beta_0(\lambda_0) > 0$. Since $\cosh sx > \sinh sx$, and $s \longrightarrow \infty$ as $\lambda \longrightarrow \Lambda_1$, this shows that $v = \beta \cosh sx - \frac{\alpha}{s}\sinh sx > 0$, when λ is near Λ_1. Again, this implies that u has no zeros in $(0,1]$. Thus we have shown that for λ near Λ_1, $N_o(\lambda) = 0$ (and $u(1,\lambda) \neq 0$, $v(1,\lambda) \neq 0$).

We shall now consider $\sigma(\lambda)$. If $\beta_1(\lambda) \equiv 0$, then $\sigma(\lambda) = 0$ by definition. Therefore we may assume $\beta_1(\lambda) > 0$.

$$p_*(\lambda)\frac{v'(1,\lambda)}{v(1,\lambda)} = p_*(\lambda)s\left[\frac{\beta \tanh s - (\alpha/s)}{\beta - (\alpha/s)\tanh s}\right].$$

Recalling that $p_*(\lambda) \geq k > 0$, it is clear that $p_*(\lambda)\frac{v'(1,\lambda)}{v(1,\lambda)} \longrightarrow \infty$ as $\lambda \longrightarrow \Lambda_1$. Since $p(1,\lambda)\frac{u'(1,\lambda)}{u(1,\lambda)} \geq p_*(\lambda)\frac{v'(1,\lambda)}{v(1,\lambda)}$, we also have $p(1,\lambda)\frac{u'(1,\lambda)}{u(1,\lambda)} \longrightarrow \infty$ as $\lambda \longrightarrow \Lambda_1$. On the other hand, by monotonicity assumption (M4), $-p(1,\lambda)\frac{\alpha_1(\lambda)}{\beta_1(\lambda)} \leq -p(1,\lambda_0)\frac{\alpha_1(\lambda_0)}{\beta_1(\lambda_0)}$, for $\lambda < \lambda_0$. Therefore, for λ near Λ_1, $p(1,\lambda)\frac{u'(1,\lambda)}{u(1,\lambda)} > -p(1,\lambda)\frac{\alpha_1(\lambda)}{\beta_1(\lambda)}$, and $u(1,\lambda)\bar{u}(\lambda) = u(1,\lambda)[\alpha_1(\lambda)u(1,\lambda) + \beta_1(\lambda)u'(1,\lambda)] > 0$.

This shows that $\sigma(\lambda) = 0$ for λ near Λ_1. Therefore $N(\lambda) = N_0(\lambda) + \sigma(\lambda) = 0$, for λ near Λ_1.

(2) We shall compare the equations

$$\text{(2.6)} \qquad \begin{cases} (p(x,\lambda_+),u')' + q(x,\lambda_+)u = 0, & \text{for } 0 \le x \le 1, \\ u(0) = \beta_0(\lambda_+), \ u'(0) = -\alpha_0(\lambda_+). \end{cases}$$

and

$$\text{(2.7)} \qquad \begin{cases} (p_*(\lambda_+)v')' + q^*(\lambda_+)v = 0, & \text{for } 0 \le x \le 1, \\ v(0) = \beta(\lambda), \ v'(0) = -\alpha, \end{cases}$$

where

$$\text{(2.8)} \qquad \alpha = p(0,\lambda_+)\alpha_0(\lambda_+), \ \beta = p_*(\lambda_+)\beta_0(\lambda_+).$$

Since $p(x,\lambda_+) \ge p_*(\lambda_+)$, $q(x,\lambda_+) \le q^*(\lambda_+)$ and $p(0,\lambda_+)\frac{\alpha_0(\lambda_+)}{\beta_0(\lambda_+)} = p_*(\lambda_+)\frac{\alpha}{\beta}$, the first and second comparison theorems apply. By assumption (L3), $\frac{q^*(\lambda_+)}{p_*(\lambda_+)} \le 0$. Therefore $\frac{q^*(\lambda_+)}{p_*(\lambda_+)} = -s^2$, where $s = \sqrt{-\frac{q^*(\lambda_+)}{p_*(\lambda_+)}}$, and equation (2.7) is equivalent to

$$\text{(2.9)} \qquad \begin{cases} v'' - s^2 v = 0 \\ v(0) = \beta, \ v'(0) = -\alpha, \end{cases}$$

whose solution is

$$\text{(2.10)} \qquad v = \beta \cosh sx - \frac{\alpha}{s} \sinh sx, \quad \text{if } s > 0,$$

or

$$\text{(2.11)} \qquad v = -\alpha x + \beta, \quad \text{if } s = 0.$$

By assumption (L3), $\alpha_0(\lambda_+)\beta_0(\lambda_+) \le 0$, which implies that $\alpha\beta \le 0$. (Of course, α and β cannot both be zero.) Therefore (2.10) and (2.11) show that $v(x)$ has no zeros in $(0,1]$. Consequently $u(x)$ has no zeros in

$[0,1]$, and $N_0(\lambda_+) = 0$.

If $s > 0$, then $p_*(\lambda_+)\frac{v'(1)}{v(1)} = p_*(\lambda_+)s\left[\frac{\beta \tanh s - (\alpha/s)}{\beta - (\alpha/s)\tanh s}\right] > 0$, since $\alpha\beta \le 0$. If $s = 0$, then $p_*(\lambda_+)\frac{v'(1)}{v(1)} = p_*(\lambda_+)\left[\frac{-\alpha}{-\alpha+\beta}\right] \ge 0$. Since $p(1,\lambda_+)\frac{u'(1)}{u(1)} \ge p_*(\lambda_+)\frac{v'(1)}{v(1)}$, this shows that $p(1,\lambda_+)\frac{u'(1)}{u(1)} \ge 0$. On the other hand, $\alpha_1(\lambda_+)\beta_1(\lambda_+) \ge 0$ by assumption (L3). This implies $-p(1,\lambda_+)\frac{\alpha_1(\lambda_+)}{\beta_1(\lambda_+)} \le 0$, so that $p(1,\lambda_+)\frac{u'(1,\lambda_+)}{u(1,\lambda_+)} \ge -p(1,\lambda_+)\frac{\alpha_1(\lambda_+)}{\beta_1(\lambda_+)}$, and $u(1,\lambda_+)\bar{u}(\lambda_+) = u(1,\lambda_+)[\alpha_1(\lambda_+)u(1,\lambda_+) + \beta_1(\lambda_+)u'(1,\lambda_+)] \ge 0$. Noting that $u(1,\lambda_+) \ne 0$, this shows that $\sigma(\lambda_+) = 0$. Therefore

$N(\lambda_+) = N_0(\lambda_+) + \sigma(\lambda_+) = 0$. Q.E.D.

Remark. By the previous lemma, if (1.1) satisfies the limit assumption (L3), then $N(\lambda_+) = 0$. If (1.1) also satisfies the monotonicity assumptions (M1) - (M4), and $\Lambda_1 < \lambda_+$, then $N(\lambda) = 0$ for $\Lambda_1 < \lambda < \lambda_+$. This will follow from Theorem 2.1 (1), which implies that $N(\lambda)$ is a nondecreasing function. Also, if $\Lambda_1 < \lambda_1$ then $N(\lambda) = 0$ for $\Lambda_1 < \lambda < \lambda_1$. This will follow from Theorem 2.1 (3). However, it can happen that $\lambda_+ = \Lambda_1 = \lambda_1$. In this case, $N(\lambda) = 1$ for λ near Λ_1. An example of this is the Sturm-Liouville problem

$$(2.12)\qquad \begin{cases} u'' + \lambda^2 u = 0, & \text{for } 0 \le x \le 1, \\ u'(0) = 0 = u'(1), \end{cases}$$

where $\Lambda_1 = 0 \le \lambda < \infty = \Lambda_2$. In this case, the first eigenvalue is $\lambda_1 = 0 = \Lambda_1$, with eigenfunction $\varphi_1(x) = 1$.

Proof of Theorem 2.1.

(1) Suppose that $\beta_1(\lambda) \not\equiv 0$. THen Lemma 2.1 implies that $N(\lambda)$ is a piecewise constant function with jump discontinuities at the points $\lambda_k : N(\lambda_k+\varepsilon) = N(\lambda_k) + 1$, $N(\lambda_k-\varepsilon) = N(\lambda_k)$. Therefore, for $\lambda' < \lambda''$ in

(Λ_1,Λ_2), $N(\lambda'')-N(\lambda')$ equals the number of jump discontinuities of $N(\lambda)$ in $[\lambda',\lambda'')$, which equals the number of eigenvalues in $[\lambda',\lambda'')$. If $\beta_1(\lambda) \equiv 0$, then the same is true, but the eigenvalues are $\lambda_k = \mu_k$.

(2) Suppose $N(\lambda') = j < k = N(\lambda'')$. By (1), there exist exactly $k-j$ eigenvalues in the interval $[\lambda',\lambda'')$. If $\beta_1(\lambda) \not\equiv 0$, then Lemma 2.1 implies that $\lambda' \le \lambda_{j+1} \le \lambda_k < \lambda''$. The same is true if $\beta_1(\lambda) \equiv 0$, where the eigenvalues are $\lambda_k = \mu_k$. Thus $\lambda' \le \lambda_{j+1} < \lambda_{j+2} < \ldots < \lambda_k < \lambda''$. (If $j+1 = k$, this sequence contains only one eigenvalue.)

(3) Lemma 2.2 implies that there exists λ_0, $\Lambda_1 \le \lambda_0 < \Lambda_2$, such that $N(\lambda_0) = 0$. (We may assume $\Lambda_1 < \lambda_0$, unless $\lambda_+ = \Lambda_1 = \lambda_1$. In this latter case, conclusion (1) remains valid if $\lambda' = \Lambda_1$.) If $\Lambda_1 < \lambda_0$, then (1) implies that there are no eigenvalues in the interval (Λ_1,λ_0), and $N(\lambda) = 0$ for $\Lambda_1 < \lambda < \lambda_0$. Thus (3) is true for $\Lambda_1 < \lambda \le \lambda_0$.

We now drop the assumption $\Lambda_1 < \lambda_0$ and assume $\lambda_0 < \lambda$. Any eigenvalues in $[\Lambda_1,\lambda)$ are contained in $[\lambda_0,\lambda)$, and the number of these eigenvalues is $N(\lambda) - N(\lambda_0) = N(\lambda)$. Q.E.D.

<u>Remark</u>. The Sturm oscillation theorem implies that if (1.1) satisfies the monotonicity assumptions (M1) - (M4) and the limit assumption (L1), then

$$\lim_{\lambda\to\Lambda_2} N(\lambda) = \infty.$$

We can now describe the shooting method.

<u>STEP 0</u>. Find values $L_0 < R_0$, such that $N(L_0) = n-1$ and $N(R_0) = n$. This implies that $L_0 \le \lambda_n < R_0$.

<u>STEP k</u>. For given values $L_{k-1} < R_{k-1}$, with $N(L_{k-1}) = n-1$, $N(R_{k-1}) = n$, find values L_k, R_k, such that $N(L_k) = n-1$, $N(R_k) = n$, $L_{k-1} \le L_k < R_k \le R_{k-1}$, and $R_k - L_k < R_{k-1} - L_{k-1}$.

STOP when $R_k - L_k \le \tau$, where τ is a given tolerance.

To implement the above steps, we need an initial value solver to compute the solution $u(x,\lambda)$ of (1.4), when $\lambda = L_k, R_k$. This allows us to calculate $N(L_k)$ and $N(R_k)$. We also need a nonlinear solver to calculate the new values L_{k+1}, R_{k+1}. We have in mind a combination of the bisection method for the integer function $N(\lambda)$, and some other method (such as the secant method) for the continuous function $\bar{u}(\lambda) = \alpha_1(\lambda)u(1,\lambda) + \beta_1(\lambda)u'(1,\lambda)$. Denoting $M_k = (L_k+R_k)/2$, the bisection method would set $L_{k+1} = L_k$, $R_{k+1} = M_k$ if $N(M_k) = n$, or $L_{k+1} = M_k$, $R_{k+1} = R_k$ if $N(M_k) = n-1$. The other part of the solver would be an iterative method to solve $\bar{u}(\lambda) = 0$. The approximate eigenvalue λ_n will be either the midpoint of the last interval $[L_k, R_k]$, or the last approximation λ_n found by the iterative method. The approximate eigenfunction $\varphi_n(x)$ will be the solution of (1.4), with $\lambda = \lambda_n$. Usually, this will have been calculated already, and no further work will be needed. STEP 0 can be carried out either by using estimates of λ_n which the user might have, or by using a related boundary value problem whose eigenvalues are known.

Assuming that the initial value problems are solved exactly, the method gives a sharp a-posteriori error estimate for the eigenvalue λ_n. Of course, the differential equations will be solved numerically. An effective implementation of this method must relate the accuracy of the initial value solver (governed by an input tolerance parameter) to the value $R_k - L_k$, and to the accuracy of the nonlinear solver for finding L_k, R_k.

If we are only interested in the eigenvalue λ_n, and not the eigenfunction $\varphi_n(x)$, then we need only concern ourselves with the count of the zeros of $u(x,\lambda)$ and the correction term σ. This can be obtained by solving various transformed formulations (such as that used in the invariant imbedding method).

The method resembles a count of the critical lengths in the invariant imbedding method (see Scott [15, Chap. 5]). However, a simple count of the critical lengths (with no correction term) does not produce a correct algorithm, for general boundary conditions. We shall return to this point in §6.

Remark. We can obtain another version of the shooting theorem if the words "increasing" and "decreasing" are interchanged in the monotonicity assumptions (M1) - (M4), while Λ_1 and Λ_2 are interchanged in the limit assumptions (L1) - (L3). In this case, the Sturm oscillation theorem would declare the existence of a decreasing sequence of eigenvalues $\lambda_m > \lambda_{m+1} > \lambda_{m+2} > \dots$, which tends to Λ_1. There is no change in the definition of $N(\lambda)$. The conclusions in the shooting theorem would be changed to the following:

(1) The interval $(\lambda', \lambda'']$ contains exactly $N(\lambda') - N(\lambda'')$ eigenvalues of (1.1).

(2) If $N(\lambda') = j > k = N(\lambda'')$, then the eigenvalues $\lambda_{k+1}, \lambda_{k+2}, \dots, \lambda_j$ exist, and $\lambda' < \lambda_j < \lambda_{j-1} < \dots < \lambda_{k+2} < \lambda_{k+1} \le \lambda''$.

(3) For $\Lambda_1 < \lambda < \Lambda_2$, (1.1) has exactly $N(\lambda)$ eigenvalues in the interval $(\lambda, \Lambda_2]$.

This version of the shooting theorem would apply to a Sturm-Liouville equation such as the acoustics problem (1.2). This new version of the theorem easily follows from the old version by considering the functions $\bar{p}(x,\lambda) = p(x,-\lambda)$, $\bar{q}(x,\lambda) = q(x,-\lambda)$, $\bar{\alpha}_1(\lambda) = \alpha_1(-\lambda)$ and $\bar{\beta}_1(\lambda) = \beta_1(-\lambda)$.

3. The Discrete Problem.

In this section, we shall discretize the boundary value problem (1.1). We shall generate a difference scheme by using finite elements. This is a convenient method which guarantees a local $O(h^2)$ error and a symmetric difference matrix. In §4, we shall apply Sturm sequences to the discrete problem. This will lead to another proof of the shooting theorem, and will account for the similarity between the shooting method and the Sturm sequence algorithm.

Recall that the energy inner product $B(u,v) = B(\lambda;u,v)$ for the problem (1.1) is given by

$$B(u,v) = -\frac{\alpha_0(\lambda)}{\beta_0(\lambda)}p(0,\lambda)u(0)v(0) + \frac{\alpha_1(\lambda)}{\beta_1(\lambda)}p(1,\lambda)u(1)v(1) + \int_0^1 (pu'v'-quv)dx.^{(*)} \tag{3.1}$$

A weak solution of (1.1) (for a fixed λ) is a function u in the Sobolev space $H^1[0,1]$, such that

$$B(u,v) = 0, \quad \text{for all} \quad v \in H^1[0,1]. \tag{3.2}$$

If (3.2) admits a nontrivial solution $u_0(x)$ for a particular value $\lambda = \lambda_0$, then λ_0 is an eigenvalue, and $u_0(x)$ is a corresponding eigenfunction. (If $\beta_0(\lambda) \equiv 0$, then $\frac{\alpha_0}{\beta_0}$ is set equal to 0 in (3.1), and $H^1[0,1]$ is replaced by the subspace of functions $v \in H^1[0,1]$, such that $v(0) = 0$. The case $\beta_1(\lambda) \equiv 0$ is treated similarly. We shall carry out the calculations in the generic case $\beta_0(\lambda) \neq 0$, $\beta_1(\lambda) \neq 0$.)

The problem will be discretized using piecewise linear functions, with

(*) Strictly speaking, the energy inner product for the operator in (1.1) is the negative of the inner product (3.1). But this is irrelevant for the equation $B(u,v) = 0$.

uniform mesh $h = 1/n$. Consider the partition $x_0 < x_1 < \dots < x_n$ of the interval $[0,1]$ by the nodes $x_i = ih$. The finite element space S_h is the space of continuous functions on $[0,1]$ which are linear on each interval $[x_{i-1}, x_i]$. The inner product (3.1) will now be restricted to S_h, and the integrals in (3.1) will be approximated by quadrature formulas. We shall use the midpoint rule for the integral $\int_0^1 pu'v'dx$, and the trapezoid rule for the integral $\int_0^1 quvdx$. This defines an inner product $B_h(u,v)$ on S_h. The finite element solution is a function $u \in S_h$, such that

$$B_h(u,v) = 0 \quad \text{for all} \quad v \in S_h. \tag{3.3}$$

A basis $v_0, v_1, \dots, v_n$ for S_h can be obtained as follows:

$$v_0(x) = \begin{cases} -x/h + 1, & \text{for } x_0 \le x \le x_1, \\ 0 & \text{elsewhere.} \end{cases} \tag{3.4a}$$

$$v_i(x) = \begin{cases} (x-x_i)/h + 1, & \text{for } x_{i-1} \le x \le x_i, \\ -(x-x_i)/h + 1, & \text{for } x_i \le x \le x_{i+1}, \\ 0 & \text{elsewhere,} \end{cases} \tag{3.4b}$$

for $1 \le i \le n-1$.

$$v_n(x) = \begin{cases} (x-x_n)/h + 1, & \text{for } x_{n-1} \le x \le x_n, \\ 0 & \text{elsewhere.} \end{cases} \tag{3.4c}$$

This basis is uniquely determined by the property

$$v_i(x_j) = \delta_{ij} \quad (0 \le i,j \le n). \tag{3.4d}$$

It also satisfies

(3.4e) The support of $v_i(x)$ is contained in the one or two intervals which contain x_i.

A function $u \in S_h$ can be expressed in the form $u(x) = \sum_{i=0}^{n} u_i v_i(x)$, where

$u_i = u(x_i)$. The equation (3.3) is now equivalent to the system of equations

$$\sum_{i=0}^{n} u_i B_h(v_i, v_j) = 0, \quad j = 0,1,\ldots,n. \tag{3.5}$$

Because of property (3.4e), $B_h(v_i, v_j) = 0$ if $|i-j| > 1$. A simple calculation shows that

$$\left\{\begin{aligned}
B_h(v_0, v_0) &= -\frac{\alpha_0}{\beta_0} p_0 + \frac{p_{\frac{1}{2}}}{h} - \frac{h}{2} q_0, \\
B_h(v_n, v_n) &= \frac{\alpha_1}{\beta_1} p_n + \frac{p_{n-\frac{1}{2}}}{h} - \frac{h}{2} q_n, \\
B_h(v_i, v_i) &= \frac{1}{h}\left(p_{i-\frac{1}{2}} + p_{i+\frac{1}{2}}\right) - hq_i, \quad 1 \le i \le n-1, \\
B_h(v_i, v_{i+1}) &= -\frac{p_{i+\frac{1}{2}}}{h}, \quad 0 \le i \le n-1.
\end{aligned}\right. \tag{3.6}$$

Here, $p_j = p_j(\lambda)$ denotes $p(jh,\lambda)$, where j is an integer or half-integer, and similarly for $q_j = q_j(\lambda)$. Multiplying the equations (3.5) by h, we obtain a system of equations:

$$A(\lambda)u = 0, \tag{3.7}$$

where $u = (u_0, u_1, \ldots, u_n)^T$ and $A(\lambda)$ is the matrix

$$A(\lambda) = \begin{bmatrix}
(b_0-a_0) & -b_0 & & & & \\
-b_0 & (b_0+b_1-a_1) & -b_1 & & & \\
 & -b_1 & (b_1+b_2-a_2) & -b_2 & & \\
 & & \ddots & \ddots & \ddots & \\
 & & & -b_{n-2} & (b_{n-2}+b_{n-1}-a_{n-1}) & -b_{n-1} \\
 & & & & -b_{n-1} & (b_{n-1}-a_n)
\end{bmatrix}. \tag{3.8}$$

The matrix coefficients here are

$$
(3.9) \qquad \begin{cases} a_0(\lambda) = \frac{h^2}{2} q_0(\lambda) + \frac{h\alpha_0(\lambda)}{\beta_0(\lambda)} p_0(\lambda), \\ a_n(\lambda) = \frac{h^2}{2} q_n(\lambda) - \frac{h\alpha_1(\lambda)}{\beta_1(\lambda)} p_n(\lambda), \\ a_i(\lambda) = h^2 q_i(\lambda), \quad \text{for} \quad 1 \le i \le n-1, \\ b_i(\lambda) = p_{i+\frac{1}{2}}(\lambda), \quad \text{for} \quad 0 \le i \le n-1. \end{cases}
$$

Recall that we have assumed (for $i = 0,1$) that either $\beta_i(\lambda) \equiv 0$, or $\beta_i(\lambda) > 0$. If $\beta_0(\lambda) \equiv 0$, then $u_0 = 0$ and the first row and column in $A(\lambda)$ are omitted. If $\beta_1(\lambda) \equiv 0$, then $u_n = 0$ and the last row and column are omitted. Thus $A(\lambda)$ is an $m \times m$ matrix, where m can be $n-1$, n or $n+1$. We shall continue to confine our calculations to the generic case $\beta_i(\lambda) > 0$, for $i = 0,1$. In this case, $A(\lambda)$ is an $(n+1)\times(n+1)$ matrix.

The standard assumptions (S1) - (S3) in §1 imply that the $a_i(\lambda)$ and $b_j(\lambda)$ are continuous functions, and $b_j(\lambda) \ge k > 0$. The monotonicity assumptions (M1) - (M4) imply that the functions $a_i(\lambda)$ are strictly increasing and the $b_j(\lambda)$ are nonincreasing.

Remark. The finite difference method leads to almost the same difference scheme as (3.7). Using the difference operator $\Delta u_i = (u_{i+\frac{1}{2}} - u_{i-\frac{1}{2}})/h$, and approximating the differential equation in (1.1) by the difference equations $\Delta(p_i \Delta u_i) + q_i u_i = 0$, we obtain all of the equations in (3.7) except the first and last equations (corresponding to the first and last rows of $A(\lambda)$). For the discretization of the boundary conditions in (1.1), a standard procedure is to discretize the derivative $u'(0)$ by $(u_1 - u_{-1})/2h$, the derivative $u'(1)$ by $(u_{n+1} - u_{n-1})/2h$, and to extend $p(x,\lambda)$ slightly outside of the interval $[0,1]$ by reflecting values about the endpoints $x = 0$ and $x = 1$. If we carry out this procedure, we obtain new first and last equations which differ by $O(h^2)$ terms from the old first and last equations in (3.7).

Therefore, we may regard the first and last equations in (3.7) as discretizations of the boundary conditions in (1.1). We may also regard the first equation in (3.7) together with the equation $u_0 = \beta_0(\lambda)$, as a discretization of the initial conditions in (1.4).

If we set $v = pu'$, then the second order equation (1.4) is converted to a first order system

$$\text{(3.10)} \qquad \begin{cases} u' = v/p, \\ v' = -qu, \\ u(0) = \beta_0, \ v(0) = -p(0)\alpha_0. \end{cases}$$

The first n equations (3.7) are equivalent to the difference scheme

$$\text{(3.11)} \qquad \begin{cases} u_i = u_{i-1} + \dfrac{h}{p_{i-\frac{1}{2}}} v_i, \\ v_i = v_{i-1} - hq_{i-1}u_{i-1}, \end{cases}$$

where $v_i = p_{i-\frac{1}{2}} \left(\dfrac{u_i - u_{i-1}}{h}\right)$. This is an implicit, general one-step method, which converges to the solution of (3.10). (See, for example, Hairer, Norsett and Wanner [9].) Therefore the discrete solution u_i of (3.7) (with the last equation omitted) converges to the continuous solution $u(x,\lambda)$ of (1.4), and the discrete derivative $u_i' = \frac{u_i - u_{i-1}}{h}$ converges to $u'(x,h)$.

4. Application of Sturm Sequences.

Let $A(\lambda)$ be the finite element matrix (3.8). The Sturm sequence $S_0(\lambda), S_1(\lambda), \ldots, S_{n+1}(\lambda)$ for $A(\lambda)$ is defined as follows. $S_0(\lambda) = 1$, and for $1 \le i \le n+1$, $S_i(\lambda)$ is the i^{th} principal minor of $A(\lambda)$:

$$
(4.1) \quad S_i(\lambda) = \begin{vmatrix}
(b_0-a_0) & -b_0 & & & & \\
-b_0 & (b_0+b_1-a_1) & -b_1 & & & \\
 & -b_1 & (b_1+b_2-a_2) & -b_2 & & \\
 & & \ddots & \ddots & \ddots & \\
 & & & -b_{i-3} & (b_{i-3}+b_{i-2}-a_{i-2}) & -b_{i-2} \\
 & & & & -b_{i-2} & (b_{i-2}+b_{i-1}-a_{i-1})
\end{vmatrix}.
$$

(In the case $i = n+1$, $b_{i-1} = b_n$ is defined to be zero.) The Sturm sequence should not be computed directly from its definition (4.1), but from the recursion relation

$$
(4.2) \quad \begin{cases}
S_{i+1} = (b_{i-1}+b_i-a_i)S_i - b_{i-1}^2 S_{i-1}, & \text{for } 1 \le i \le n-1, \\
S_{n+1} = (b_{n-1}-a_n)S_n - b_{n-1}^2 S_{n-1}, &
\end{cases}
$$

which follows from the cofactor expansion of the determinant S_{i+1} by its last row. Note that $S_{n+1} = \det[A(\lambda)]$. The main significance of the Sturm sequence is that it can be used to approximate the zeros of $\det[A(\lambda)]$ (which, in turn, approximate the eigenvalues of (1.1)). See Stoer, Bulirsch [17] for the application of Sturm sequences to linear eigenvalue problems.

Definition 4.1. For a given λ in (Λ_1, Λ_2), $c(\lambda)$ is the number of sign changes in the sequence $S_0(\lambda), S_1(\lambda), \ldots, S_{n+1}(\lambda)$, after the zero terms (if any) have been omitted from the sequence.

The following theorem, which is proved in Greenberg [7, §4], gives the main facts about the Sturm sequence (4.1).

<u>Theorem 4.1</u>. Let $A(\lambda)$ be the finite element matrix in (3.8). Suppose that (1.1) satisfies the monotonicity assumptions (M1) - (M4). Then:

(1) For any numbers $\lambda' < \lambda''$ in (Λ_1, Λ_2), $\det[A(\lambda)]$ has exactly $c(\lambda'') - c(\lambda')$ different zeros in the interval $[\lambda', \lambda'')$.

(2) The zeros of $S_i(\lambda)$ and $S_{i-1}(\lambda)$ are interlaced, for $1 \le i \le n+1$.

If (1.1) also satisfies either the limit assumption (L2) or (L3), then:

(3) For any λ_0 in (Λ_1, Λ_2), $\det[A(\lambda)]$ has exactly $c(\lambda_0)$ different zeros in the interval $(\Lambda_1, \lambda_0)^{(*)}$.

If (1.1) satisfies assumptions (M1) - (M4), (L1) and either (L2) or (L3), then:

(4) $\det[A(\lambda)]$ has exactly $n+1$ different zeros in $(\Lambda_1, \Lambda_2)^{(*)}$.

We shall now use Sturm sequences to study the discrete problem. The initial value problem (1.4) is discretized by the first n equations of the system (3.7), together with the initial condition $u_0 = \beta_0(\lambda)$. Let u_i $(0 \le i \le n)$ denote the solution of the discrete initial value problem.

<u>Theorem 4.2</u>. $u_i = t_i S_i$ (for $0 \le i \le n$), where $t_i = \beta_0/(b_{i-1} b_{i-2} \ldots b_0)^{(**)}$.

(*) If $\lambda_+ = \Lambda_1$ in limit assumption (L3), then the interval (Λ_1, λ_0) should be replaced by $[\Lambda_1, \lambda_0)$ in conclusion (3), and (Λ_1, Λ_2) replaced by $[\Lambda_1, \Lambda_2)$ in conclusion (4).

(**) This assumes that $\beta_0(\lambda) > 0$. See the remark following the theorem, for the case $\beta_0(\lambda) \equiv 0$.

Proof. Consider the first $(i+1)$ equations of (3.7):

$$(4.3)\quad \begin{cases} (b_0-a_0)u_0 - b_0u_1 & = 0 \\ -b_0u_0 + (b_0+b_1-a_1)u_1 - b_1u_2 & = 0 \\ \quad \cdot \quad \cdot \quad \cdot & \\ -b_{i-2}u_{i-2} + (b_{i-2}+b_{i-1}-a_{i-1})u_{i-1} - b_{i-1}u_i & = 0 \\ -b_{i-1}u_{i-1} + (b_{i-1}+b_i-a_i)u_i & = b_iu_{i+1}. \end{cases}$$

We can solve for u_i using Cramer's rule, to obtain $u_i = \dfrac{b_iu_{i+1}S_i}{S_{i+1}}$, which can be written:

$$(4.4)\qquad \frac{u_{i+1}}{S_{i+1}} = \frac{1}{b_i}\frac{u_i}{S_i}.$$

First suppose that all $S_j \neq 0$ and all $u_j \neq 0$. Letting $u_i/S_i = t_i$, (4.4) shows that $t_{i+1} = t_i/b_i$. Thus $t_i = t_{i-1}/b_{i-1} = t_{i-2}/(b_{i-1}b_{i-2}) = \ldots = t_0/(b_{i-1}b_{i-2}\ldots b_0)$. Since $u_0 = \beta_0$ and $S_0 = 1$, $t_0 = \beta_0$ and $t_i = \beta_0/(b_{i-1}b_{i-2}\ldots b_0)$.

Now consider the case where some u_j or S_j are zero. Note that two consecutive u_j cannot be zero, and two consecutive S_j cannot be zero. This follows from the recursion relations (4.2) and

$$(4.5)\qquad -b_{i-1}u_{i-1} + (b_{i-1}+b_i-a_i)u_i - b_iu_{i+1} = 0.$$

If $u_i = 0 = u_{i+1}$, then (4.5) implies that $u_{i-1} = 0$. (Recall that $b_{i-1} = p_{i-\frac{1}{2}} \neq 0$). Similarly, $0 = u_{i-1} = u_{i-2} = \ldots = u_0$, which contradicts $u_0 = \beta_0 \neq 0$. In the same way, (4.2) implies that two consecutive S_j are not zero. Equation (4.4) may be written $u_iS_{i+1} = b_iu_{i+1}S_i$, which shows that $u_i = 0$ if and only if $S_i = 0$. If $u_i = S_i = 0$, then equations (4.2) and (4.5) imply

$$(4.6) \qquad u_{i+1} = -\frac{b_{i-1}}{b_i}u_{i-1}, \quad S_{i+1} = -b_{i-1}^2 S_{i-1}.$$

Therefore $\frac{u_{i+1}}{S_{i+1}} = \frac{1}{b_i b_{i-1}} \frac{u_{i-1}}{S_{i-1}}$, so that $t_{i+1} = t_{i-1}/(b_i b_{i-1})$, even if $u_i = S_i = 0$. Q.E.D.

Remark 1. In Theorem 4.2, we have assumed that $\beta_0(\lambda) > 0$. If $\beta_0(\lambda) \equiv 0$, then the first equation in (3.7) is omitted, $u_0 = 0$, and

$$(4.7) \quad S_i = \begin{vmatrix} (b_0+b_1-a_1) & -b_1 & & & \\ -b_1 & (b_1+b_2-a_2) & -b_2 & & \\ & \ddots & \ddots & \ddots & \\ & & -b_{i-2} & (b_{i-2}+b_{i-1}-a_{i-1}) & -b_{i-1} \\ & & & -b_{i-1} & (b_{i-1}+b_i-a_i) \end{vmatrix}.$$

In this case (4.4) is replaced by

$$(4.8) \qquad \frac{u_{i+1}}{S_i} = \frac{1}{b_i} \frac{u_i}{S_{i-1}},$$

and so $u_i = t_i S_{i-1}$ (for $1 \le i \le n$), where $t_i = t_1/(b_{i-1}b_{i-2}\cdots b_1)$, and $t_1 = u_1 \neq 0$. Thus, if $\beta_0 = 0$, then all t_i have the same sign as u_1. If $\beta_0 > 0$, then Theorem 4.2 shows that all $t_i > 0$. In both cases, the solution $(u_0, u_1, \ldots, u_n)$ of the discrete initial value problem has the same number of sign changes as the Sturm sequence $S_0, S_1, \ldots, S_n$. (Note that if $\beta_0 > 0$, there is an additional function S_{n+1}, which has not been included in the last sequence.)

Remark 2. Theorem 4.2 indicates that we can expect overflow to occur, if we attempt to calculate S_i for small h. For u_i can be expected to have moderate size, since it is the discrete solution of the initial value problem

(1.4). But $S_i = u_i/t_i$, where $1/t_i = \left(\frac{1}{\beta_0}\right) b_0 b_1 \ldots b_{i-1} = \left(\frac{1}{\beta_0}\right) p_{\frac{1}{2}} p_{\frac{3}{2}} \ldots p_{i-\frac{1}{2}}$

$\approx \left(\frac{1}{\beta_0}\right) e^{\frac{1}{h}\int_0^{x_i} \ln p(x)dx}$. This observation illustrates how how the closure method can give insight into a numerical process.

Note that the matrix $A(\lambda)$ and all the functions derived from it depend on the mesh size h. We shall now exhibit this dependence explicitly:

$A(\lambda) = A(h,\lambda)$, $S_i = S_i(h,\lambda)$, $c(\lambda) = c(h,\lambda)$, $u_i = u_i(h,\lambda)$.

Theorem 4.3. Suppose that (1.1) satisfies the monotonicity assumptions (M1) - (M4), $\Lambda_1 < \lambda < \Lambda_2$(*), and λ is not an eigenvalue of (1.1). Then $\lim_{h \to 0} c(h,\lambda) = N(\lambda)$.

Proof. We shall assume that $\beta_0(\lambda) > 0$. The case $\beta_0(\lambda) \equiv 0$ is similar. (See the remark at the end of the proof.)

Let $c_o(h,\lambda)$ be the number of sign changes in the sequence $S_0(h,\lambda), S_1(h,\lambda), \ldots, S_n(h,\lambda)$. (Note that the last function S_{n+1} has been omitted.) By Theorem 4.2, $c_o(h,\lambda)$ equals the number of sign changes in the sequence $u_0, u_1, \ldots, u_n$ (because all $t_i > 0$). If $u(1,\lambda) \neq 0$, then, as $h \to 0$, $c_o(h,\lambda)$ approaches the number of zeros of $u(x,\lambda)$ in $(0,1)$. Thus $\lim_{h \to 0} c_o(h,\lambda) = N_o(\lambda)$. (The possibility $u(1,\lambda) = 0$ will be considered below.)

We shall now look for the relation between the last function S_{n+1} and the correction term $\sigma(\lambda)$. By the recursion relation (4.2), $S_{n+1} = (b_{n-1} - a_n)S_n - b_{n-1}^2 S_{n-1}$. Using the relations $u_i = t_i S_i$ and $t_{i+1} = t_i/b_i$ from Theorem 4.2, we obtain

(*) If the coefficient functions in (1.1) can be extended continuously to $\lambda = \Lambda_1$, then Theorem 4.3 is valid for $\Lambda_1 \leq \lambda < \Lambda_2$.

$$S_{n+1} = (b_{n-1}-a_n)u_n/t_n - b_{n-1}^2 u_{n-1}/t_{n-1}$$

$$= \frac{1}{t_n}\,[(b_{n-1}-a_n)u_n - b_{n-1}u_{n-1}],$$

or

$$t_n S_{n+1} = (b_{n-1}-a_n)u_n - b_{n-1}u_{n-1}.$$

Using the formulas (3.9) for b_{n-1}, a_n, this becomes

$$t_n S_{n+1} = (p_{n-\frac{1}{2}} + h\frac{\alpha_1}{\beta_1}p_n - \frac{h^2}{2}q_n)u_n - p_{n-\frac{1}{2}}u_{n-1}$$

$$= h\frac{\alpha_1}{\beta_1}p_n u_n + p_{n-\frac{1}{2}}(u_n - u_{n-1}) - \frac{h^2}{2}q_n u_n$$

$$= \frac{hp_n}{\beta_1}[\alpha_1 u_n + \beta_1\left(\frac{u_n-u_{n-1}}{h}\right)] - (p_n - p_{n-\frac{1}{2}})(u_n - u_{n-1}) - \frac{h^2}{2}q_n u_n.$$

Letting $u_n' = \frac{u_n-u_{n-1}}{h}$, we have shown

$$t_n S_{n+1} = \frac{hp_n}{\beta_1}[\alpha_1 u_n + \beta_1 u_n'] + O(h^2). \tag{4.9}$$

Recall that $\bar{u}(\lambda) = \alpha_1(\lambda)u(1,\lambda) + \beta_1(\lambda)u'(1,\lambda)$. Clearly $\alpha_1 u_n + \beta_1 u_n' \to \bar{u}$ as $h \to 0$. Since we have assumed that λ is not an eigenvalue of (1.1), $\bar{u}(\lambda) \neq 0$. Equation (4.9) shows that $\operatorname{sgn} S_{n+1} = \operatorname{sgn} \bar{u}(\lambda)$, for small h. We shall now consider two cases, according to the possibilities that $u(1,\lambda) = 0$ or not.

<u>Case 1</u>. $u(1,\lambda) \neq 0$.

For small h, $c_o(h,\lambda) = N_o(\lambda)$, $\operatorname{sgn} S_{n+1} = \operatorname{sgn} \bar{u}(\lambda)$ and $\operatorname{sgn} S_n = \operatorname{sgn} u_n = \operatorname{sgn} u(1,\lambda)$. Thus $\operatorname{sgn} S_n S_{n+1} = \operatorname{sgn} u(1,\lambda)\bar{u}(\lambda)$. Referring to Definition 2.1, this shows that $\sigma(\lambda) = 0$ if $S_n S_{n+1} > 0$, and $\sigma(\lambda) = 1$ if $S_n S_{n+1} < 0$. Therefore $c(h,\lambda) = N(\lambda)$, for small h.

Case 2. $u(1,\lambda) = 0$.

In this case, $\sigma(\lambda) = 1$ and $N(\lambda) = N_o(\lambda) + 1$. Since $u(1,\lambda) = 0$, $u'(1,\lambda) \neq 0$. We shall suppose that $u'(1,\lambda) > 0$. (The case $u'(1,\lambda) < 0$ is similar.) Since $\bar{u}(\lambda) = \beta_1(\lambda)u'(1,\lambda)$, and $\bar{u}(\lambda) \neq 0$, it follows that $\beta_1(\lambda) > 0$ and $\bar{u}(\lambda) > 0$.

For small h, the approximate solution $(u_0, u_1, \ldots, u_n)$ will have $N_o(\lambda)$ sign changes corresponding to the interior zeros of $u(x,\lambda)$. (Since $u(1,\lambda) = 0$, there may also be a spurious sign change near the end of the sequence.) The $N_o(\lambda)$ sign changes will occur in an initial subsequence $u_0, u_1, \ldots, u_i$, where the node ih is near the largest interior zero of $u(x,\lambda)$.

Since $u'(1,\lambda) > 0$ and $u(1,\lambda) = 0$, there is a point x_0 in $(0,1)$, such that $u(x,\lambda) < 0$ for $x_0 \le x < 1$, and $u'(x,\lambda) > 0$ for $x_0 \le x \le 1$. Correspondingly, the tail end of the solution sequence will be increasing: $u_j < u_{j+1} < \ldots < u_n$ (where $i < j$). Furthermore, there are no sign changes in the subsequence $u_i, u_{i+1}, \ldots, u_j$, and $u_j < 0$. But it is possible that $u_n \ge 0$. This means that a spurious sign change may occur in the tail end of the sequence. Of course, all of these things occur in the Sturm sequence, since $S_m = u_m/t_m$. Thus, there are $N_o(\lambda)$ sign changes in the subsequence $S_0, S_1, \ldots, S_i$; there are no sign changes in the subsequence $S_i, S_{i+1}, \ldots, S_j$; $S_j < S_{j+1} < \ldots < S_n$; and $S_j < 0$. Also $S_{n+1} > 0$, since $\operatorname{sgn} S_{n+1} = \operatorname{sgn} \bar{u}(\lambda)$, for small h.

We now have two possibilities: either $S_n < 0$ or $S_n \ge 0$. If $S_n < 0$, then a sign change occurs between S_n and S_{n+1}. Therefore, $c(h,\lambda) = N_o(\lambda) + 1 = N(\lambda)$. If $S_n \ge 0$, then a single sign change occurs in the subsequence $S_j, S_{j+1}, \ldots, S_{n+1}$. Again $c(h,\lambda) = N_o(\lambda) + 1 = N(\lambda)$. This concludes Case 2.

Until now, we have assumed that $\beta_0(\lambda) > 0$. If $\beta_0(\lambda) \equiv 0$, then we must delete the first row and column of the matrix $A(\lambda)$ in (3.8), and set $u_0 = 0$. Theorem 4.2 is changed slightly as indicated in Remark 1. After these modifications, the proof is the same as above. Q.E.D.

Remark 3. In Theorem 4.3, λ is not allowed to be an eigenvalue of (1.1). If $\lambda = \lambda_k$ is an eigenvalue, then for small $\varepsilon > 0$, and small h, the discrete problem has a unique eigenvalue $\bar{\lambda}_k$ in $(\lambda_k-\varepsilon, \lambda_k+\varepsilon)$. (See Theorem A.1 in the appendix.) If we do not use quadratures in the finite element discretization, then $\lambda_k \le \bar{\lambda}_k$. If this were also true with quadratures, then Theorem 4.3 would be valid for $\lambda = \lambda_k$. Unfortunately, this does not seem to be true, in general. Moreover, if we were to use a pure finite element discretization without quadratures, then the matrix $A(\lambda)$ in (3.8) would not have the structure necessary for the application of Sturm sequences.

Lemma 4.1.

(1) Let $\Lambda_1 < \lambda' < \lambda'' < \Lambda_2$. If the interval $[\lambda', \lambda'']$ contains no eigenvalues of (1.1), then $N(\lambda') = N(\lambda'')$.

(2) If λ_k is an eigenvalue of (1.1), then for small $\varepsilon > 0$, $N(\lambda_k-\varepsilon) = N(\lambda_k)$ and $N(\lambda_k+\varepsilon) = N(\lambda_k) + 1$.

Proof. (1) For small h, the discrete problem has no eigenvalues in $[\lambda', \lambda'']$. (See the appendix for properties of the approximating eigenvalues $\bar{\lambda}_k$.) Theorem 4.3 implies that $c(h,\lambda) = N(\lambda')$ and $c(h,\lambda'') = N(\lambda'')$, for small h. Theorem 4.1 (1) implies that $c(h,\lambda') = c(h,\lambda'')$, and therefore $N(\lambda') = N(\lambda'')$.

(2) We shall consider two cases, according to the possibilities that $u(1,\lambda_k) = 0$ or not.

$\underline{\text{Case 1}}$. $u(1,\lambda_k) \neq 0$.

In this case, the number of zeros of $u(x,\lambda)$ in $(0,1)$ does not change if λ is moved slightly away from λ_k. Therefore $N_o(\lambda_k-\varepsilon) = N_o(\lambda_k) = N_o(\lambda_k+\varepsilon)$. By definition, $\sigma(\lambda_k) = 0$, so $N(\lambda_k) = N_o(\lambda_k)$. For small h, the discrete problem has a unique eigenvalue $\bar{\lambda}_k$ in the interval $(\lambda_k-\varepsilon, \lambda_k+\varepsilon)$ (see the appendix). Therefore $c(h,\lambda_k+\varepsilon) = c(h,\lambda_k-\varepsilon)+1$. Theorem 4.3 implies that $N(\lambda_k+\varepsilon) = N(\lambda_k-\varepsilon)+1$. This implies that $\sigma(\lambda_k+\varepsilon) = 1$ and $\sigma(\lambda_k-\varepsilon) = 0$, so $N(\lambda_k-\varepsilon) = N(\lambda_k)$ and $N(\lambda_k+\varepsilon) = N(\lambda_k)+1$.

$\underline{\text{Case 2}}$. $u(1,\lambda_k) = 0$.

In this case, $\beta_1(\lambda) = 0$, $\sigma(\lambda) = 0$ and $N(\lambda) = N_o(\lambda)$, for $\Lambda_1 < \lambda < \Lambda_2$. When λ is moved slightly from λ_k to $\lambda_k \pm \varepsilon$, the number of zeros of $u(x,\lambda)$ in $(0,1)$ either remains the same, or increases by 1. Thus each of the numbers $N_o(\lambda_k+\varepsilon)$, $N_o(\lambda_k-\varepsilon)$ equals either $N_o(\lambda_k)$ or $N_o(\lambda_k)+1$. As in case 1, $c(h,\lambda_k+\varepsilon) = c(h,\lambda_k-\varepsilon)+1$, for small h, and therefore $N_o(\lambda_k+\varepsilon) = N_o(\lambda_k-\varepsilon)+1$. This implies that $N_o(\lambda_k-\varepsilon) = N_o(\lambda_k)$ and $N_o(\lambda_k+\varepsilon) = N_o(\lambda_k)+1$. Q.E.D.

$\underline{\text{Lemma 4.2}}$.

(1) If (1.1) satisfies the limit assumption (L3), then $S_i(h,\lambda_+) \geq 0$ for all $h = 1/n$, and $1 \leq i \leq n+1$.

(2) If (1.1) satisfies the limit assumption (L2), then there exists $h_0 > 0$ and λ_0 in (Λ_1, Λ_2), such that $S_i(h,\lambda_0) \geq 0$ for all $h = 1/n < h_0$, and $1 \leq i \leq n+1$.

$\underline{\text{Proof}}$. (1) In Greenberg [7, Lemma 3.3] it is shown for a matrix A of type (3.8), that if $b_i \geq 0$ for $0 \leq i \leq n-1$, and $a_i \leq 0$ for $0 \leq i \leq n$, then $\det[A] \geq 0$. Referring to (3.9), we see that this implies (1).

(2) Assumption (L2) implies that $\lim_{\lambda \to \Lambda_1} q^*(\lambda) = -\infty$. Therefore there is a λ_0 in (Λ_1, Λ_2) such that $q^*(\lambda_0) < 0$. Referring to (3.9), we see that $a_i(h,\lambda_0) < 0$ for all $h = 1/n$, and $1 \le i \le n-1$, while $b_i(h,\lambda) > 0$ for all λ, $h = 1/n$ and $0 \le i \le n-1$. We must still deal with a_0 and a_n. (Recall that if $\beta_0(\lambda) \equiv 0$, then the first row and column of $A(\lambda)$ in (3.8) are omitted, so a_0 does not occur. Similarly, if $\beta_1(\lambda) \equiv 0$, then a_n does not occur. Therefore we may assume that $\beta_0(\lambda) > 0$ and $\beta_1(\lambda) > 0$.)

Define $\bar{a}_0(h,\lambda) = \frac{h^2}{2} q_0(\lambda)$, $\bar{b}_0(h,\lambda) = p_{\frac{1}{2}}(\lambda) - h\frac{\alpha_0(\lambda)}{\beta_0(\lambda)} p_0(\lambda)$, $\bar{a}_n(h,\lambda) = \frac{h^2}{2} q_n(\lambda)$ and $\bar{b}_{n-1}(h,\lambda) = p_{n-\frac{1}{2}})(\lambda) + h\frac{\alpha_1(\lambda)}{\beta_1(\lambda)} p_n(\lambda)$. Then $\bar{a}_0(h,\lambda_0) < 0$, $\bar{a}_n(h,\lambda_0) < 0$, $\bar{b}_0 - \bar{a}_0 = b_0 - a_0$ and $\bar{b}_{n-1} - \bar{a}_n = b_{n-1} - a_n$. We must show that $\bar{b}_0(h,\lambda_0) > 0$ and $\bar{b}_{n-1}(h,\lambda_0) > 0$ for small h. Let $m = \max_{0 \le x \le 1} p(x,\lambda_0)$, and recall that $p(x,\lambda) \ge k > 0$, by assumption (S2). Then $\bar{b}_0(h,\lambda_0) \ge k - h\left|\frac{\alpha_0(\lambda_0)}{\beta_0(\lambda_0)}\right| m$ and $\bar{b}_{n-1}(h,\lambda_0) \ge k - h\left|\frac{\alpha_1(\lambda_0)}{\beta_1(\lambda_0)}\right| m$. Let $h_0 = \min(\frac{k}{m}\left|\frac{\beta_0(\lambda_0)}{\alpha_0(\lambda_0)}\right|, \frac{k}{m}\left|\frac{\beta_1(\lambda_0)}{\alpha_1(\lambda_0)}\right|)$. Then $\bar{b}_0(h,\lambda_0) > 0$ and $\bar{b}_{n-1}(h,\lambda_0) > 0$, for $0 < h < h_0$. Q.E.D.

<u>A Second Proof of the Shooting Theorem</u>.

(1) Suppose that (1.1) satisfies the monotonicity assumptions (M1)-(M4). Lemma 4.1 shows that $N(\lambda)$ is a piecewise constant function, with jump discontinuities at the eigenvalues λ_k, and $N(\lambda_k - \varepsilon) = N(\lambda_k)$, $N(\lambda_k + \varepsilon) = N(\lambda_k) + 1$. Therefore $N(\lambda'') - N(\lambda')$ equals the number of jump discontinuities in $[\lambda', \lambda'')$, which equals the number of eigenvalues in $[\lambda', \lambda'')$.

(2) By definition of the eigenvalue λ_k, $u(x,\lambda_k)$ has exactly $k-1$ interior zeros. By definition of $N(\lambda)$, $N(\lambda_k) = k-1$. For $\lambda_{k-1} < \lambda \le \lambda_k$, $N(\lambda) = N(\lambda_k)$ because the interval $[\lambda, \lambda_k)$ contains no eigenvalues of (1.1). Thus $N(\lambda) = k-1$ for $\lambda_{k-1} < \lambda \le \lambda_k$. Now suppose that $N(\lambda') = j < k < N(\lambda'')$. Then the interval $[\lambda', \lambda'')$ contains exactly $k-j$

eigenvalues and $\lambda' \le \lambda_{j+1} < \dots < \lambda_k < \lambda''$.

(3) Suppose that (1.1) also satisfies either limit assumption (L2) or (L3). By Lemma 4.2, there exists λ_0 such that $S_i(h,\lambda_0) \ge 0$ for small $h = 1/n$, and $1 \le i \le n+1$. This implies that $c(h,\lambda_0) = 0$. If λ_0 is not an eigenvalue of (1.1), then $c(h,\lambda_0) = N(\lambda_0)$, for small h, and therefore $N(\lambda_0) = 0$.

Suppose that λ_0 is an eigenvalue of (1.1). For small h, the discrete problem has a unique eigenvalue $\bar{\lambda}_0$ in $(\lambda_0-\varepsilon,\lambda_0+\varepsilon)$ (or in $[\lambda_0,\lambda_0+\varepsilon)$, if $\lambda_0 = \Lambda_1$). Theorem 4.1 (1) implies that $c(h,\lambda_0+\varepsilon) = c(h,\lambda_0-\varepsilon)+1$. But since $c(h,\lambda)$ is a nondecreasing function of λ, and since $c(h,\lambda_0) = 0$, it follows that $c(h,\lambda_0-\varepsilon) = 0$. Therefore $c(h,\lambda_0+\varepsilon) = 1$, for small h. Since $c(h,\lambda_0+\varepsilon) = N(\lambda_0+\varepsilon)$, this implies that $N(\lambda_0+\varepsilon) = 1$. By Lemma 4.1, $N(\lambda_0+\varepsilon) = N(\lambda_0)+1$, therefore $N(\lambda_0) = 0$. Now (1) implies that (1.1) has no eigenvalues in (Λ_1,λ_0), and $N(\lambda) = 0$ for $\Lambda_1 < \lambda \le \lambda_0$. If $\lambda > \lambda_0$, then any eigenvalues in $[\Lambda_1,\lambda)$ are contained in $[\lambda_0,\lambda)$, and the number of these eigenvalues is $N(\lambda) - N(\lambda_0) = N(\lambda)$. Q.E.D.

5. Sturmian Theorems for the Discrete Problem.

In this section, we shall prove discrete versions of the Sturm comparison, oscillation and separation theorems. For $1 \le i \le n$, let $A_i(\lambda)$ be the $i \times i$ matrix which is the upper left corner of the matrix $A(\lambda)$ in (3.8):

$$(5.1) \quad A_i(\lambda) = \begin{bmatrix} (b_0-a_0) & -b_0 & & & & \\ -b_0 & (b_0+b_1-a_1) & -b_1 & & & \\ & -b_1 & (b_1+b_2-a_2) & -b_2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -b_{i-3} & (b_{i-3}+b_{i-2}-a_{i-2}) & -b_{i-2} \\ & & & & -b_{i-2} & (b_{i-2}+b_{i-1}-a_{i-1}) \end{bmatrix}.$$

$S_i = \det[A_i(\lambda)]$ is the i^{th} function in the Sturm sequence for $A(\lambda)$. The Sturm sequence for $A_i(\lambda)$ is $S_0, S_1, S_2, \ldots, S_i$.

Definition 5.1. For $1 \le i \le n$, and for a given λ in (Λ_1, Λ_2), $c_i(\lambda)$ is the number of sign changes in the sequence $S_0(\lambda), S_1(\lambda), \ldots, S_i(\lambda)$, after the zero terms (if any) have been omitted from the sequence.

(Note that $c_i(\lambda)$ has been defined for $1 \le i \le n$. We shall not use the notation $c_{n+1}(\lambda)$ for $c(\lambda)$, or $A_{n+1}(\lambda)$ for $A(\lambda)$.) The following theorem is proved in Greenberg [7, Theorem 3.2].

Theorem 5.1. Suppose that

(1) $a_j(\lambda)$ is a strictly increasing function, for $0 \le j \le i-1$,

(2) $b_j(\lambda)$ is a nonincreasing function, for $0 \le j \le i-1$,

(3) $b_j(\lambda)$ has no zeros, for $1 \le j \le i-2$.

Let $\lambda' < \lambda''$ be numbers in (Λ_1, Λ_2). Then $\det[A_i(\lambda)]$ has exactly $c_i(\lambda'') - c_i(\lambda')$ different zeros in the interval $[\lambda', \lambda'')$.

In the present situation, where $A(\lambda)$ is the finite element matrix corresponding to a discretization of (1.1), $b_j = p_{j+\frac{1}{2}} > 0$, so condition (3) in the theorem is satisfied. If (1.1) satisfies the monotonicity assumptions (M1) - (M3), then conditions (1) and (2) are also satisfied. Note that assumption (M4) is not required, because $\alpha_1(\lambda)$ and $\beta_1(\lambda)$ appear only in the last row of $A(\lambda)$, so they do not appear in any of the matrices $A_i(\lambda)$, $1 \le i \le n$.

Let $B = B(\lambda)$ be the $n\times(n+1)$ matrix obtained from $A(\lambda)$ by deleting the last row:

$$(5.2)\quad B = \begin{bmatrix} (b_0-a_0) & -b_0 & & & & \\ -b_0 & (b_0+b_1-a_1) & -b_1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & -b_{n-3} & (b_{n-3}+b_{n-2}-a_{n-2}) & -b_{n-2} & \\ & & & -b_{n-2} & (b_{n-2}+b_{n-1}-a_{n-1}) & -b_{n-1} \end{bmatrix}.$$

Let $u = u(\lambda) = (u_0(\lambda), u_1(\lambda), \ldots, u_n(\lambda))^T$ be the solution of the discrete version of the unitial value problem (1.4). In other words, u is the solution of

$$(5.3)\qquad Bu = 0,\ u_0 = \beta_0(\lambda).$$

We shall usually assume that $\beta_0(\lambda) > 0$, with an occasional remark on the case $\beta_0(\lambda) \equiv 0$. Although B and u depend on the mesh size h as well as λ, we shall usually suppress h from the notation, since the mesh size will usually be fixed, in this section.

The last $n-1$ equations of (5.3) are

(5.4) $\quad -b_{i-1}u_{i-1} + (b_{i-1}+b_i-a_i)u_i - b_iu_{i+1} = 0, \quad \text{for} \quad 1 \le i \le n-1.$

This implies that two consecutive terms u_i, u_{i+1} cannot both be zero. (We have already discussed this after equation (4.5), which coincides with (5.4).) Also, (5.4) implies that if $u_i = 0$, then $u_{i-1}u_{i+1} < 0$ (since all $b_j > 0$).

Definition 5.2. We shall say that $u(\lambda)$ has a sign change at $u_j(\lambda)$ if either $u_{j-1}(\lambda)u_j(\lambda) < 0$, or $u_{j-1}(\lambda) = 0$.
Note that if $u_{j-1} = 0$, then $j-1 > 0$ and $u_{j-2}u_j < 0$.

Theorem 5.2 (First Comparison Theorem). Suppose that (1.4) satisfies the monotonicity conditions (M1) - (M3). Let $\lambda_1 < \lambda_2$ be numbers in (Λ_1, Λ_2). Then:

(1) $u(\lambda_2)$ has at least as many sign changes as $u(\lambda_1)$. Furthermore, $u(\lambda_2)$ has more sign changes than $u(\lambda_1)$ if and only if $u_n(\lambda)$ has a zero in the interval $[\lambda_1, \lambda_2)$.

(2) If the i^{th} sign change for $u(\lambda_1)$ occurs at $u_{j_1}(\lambda_1)$, and for $u(\lambda_2)$ at $u_{j_2}(\lambda_2)$, then $j_2 \le j_1$. Furthermore, $j_2 < j_1$ if and only if $u_{j_1}(\lambda)$ has a zero in (λ_1, λ_2).

Proof.

(1) By Theorem 4.2, $u(\lambda)$ has the same number of sign changes as the Sturm sequence $S_0(\lambda), S_1(\lambda), \ldots, S_n(\lambda)$ for $A_n(\lambda)$. Therefore $u(\lambda)$ has exactly $c_n(\lambda)$ sign changes. By Theorem 5.1, $c_n(\lambda_1) \le c_n(\lambda_2)$. Furthermore, $c_n(\lambda_1) < c_n(\lambda_2)$ if and only if $S_n(\lambda) = u_n(\lambda)/t_n(\lambda)$ has a zero in $[\lambda_1, \lambda_2)$.

(2) Similarly, the sequence $u_0(\lambda), u_1(\lambda), \ldots, u_j(\lambda)$ has exactly $c_j(\lambda)$ sign changes. By assumption, $c_{j_1}(\lambda_1) = i$. Since $c_{j_1}(\lambda_1) \le c_{j_1}(\lambda_2)$, the sequence $u_0(\lambda_2), u_1(\lambda_2), \ldots, u_{j_1}(\lambda_2)$ has at least i sign changes. Therefore

$j_2 \leq j_1$. Furthermore, $j_2 < j_1$ if and only if $u_0(\lambda_2), u_1(\lambda_2), \ldots, u_{j_1}(\lambda_2)$ has more than i sign changes. The latter is true if and only if $u_{j_1}(\lambda)$ has a zero in $[\lambda_1, \lambda_2)$. However, since a sign change occurs at $u_{j_1}(\lambda_1)$, Definition 5.2 implies that $u_{j_1}(\lambda_1) \neq 0$. Therefore the zero must occur in (λ_1, λ_2). Q.E.D.

Remark. In Greenberg [7], Sturm sequences are discussed from an axiomatic standpoint. It is noted there that one of the axioms implies the following fact: If $S_{i+1}(\lambda_0) = 0$, then $S_i(\lambda)S_{i+1}(\lambda) > 0$ for $\lambda_0 - \varepsilon < \lambda < \lambda_0$, and $S_i(\lambda)S_{i+1}(\lambda) < 0$ for $\lambda_0 < \lambda < \lambda_0 + \varepsilon$. Thus, as λ increases past λ_0, a sign change is generated between $S_i(\lambda)$ and $S_{i+1}(\lambda)$. Of course, the same is true for $u_i(\lambda)$ and $u_{i+1}(\lambda)$. If $u_i(\lambda)u_{i+1}(\lambda) < 0$, then as λ increases, $u_{i+1}(\lambda)$ cannot change sign before $u_i(\lambda)$ does. This means that, as λ increases, the sign changes move from right to left. Also, if $u_{i-1}(\lambda)u_i(\lambda) > 0$ and $u_i(\lambda)u_{i+1}(\lambda) > 0$, then $u_i(\lambda)$ cannot have a sign change before both $u_{i-1}(\lambda)$ and $u_{i+1}(\lambda)$. For if $u_i(\lambda_0) = 0$, then $u_{i-1}(\lambda_0)u_{i+1}(\lambda_0) < 0$. This means that sign changes cannot appear spontaneously in the middle of the sequence. They must start at $u_n(\lambda)$, and move to the left. This is exactly what happens to the zeros of the solution $u(x, \lambda)$ of the continuous problem (1.4).

The following lemma is proved in Greenberg [7, Lemma 3.2]. (There is a slight difference in notation, for two reasons. First of all, the first row of $A(\lambda)$ is here indexed by $i = 0$, while in [7], it is indexed by $i = 1$. Second, we assume here that none of the b_j have zeros, while in [7], this assumption is made only for the first $n - 1$ values b_j.)

Lemma 5.1. Suppose that, for $0 \leq j \leq n - 1$,

(1) $a_j(\lambda)$ is a strictly increasing function,

(2) $b_j(\lambda)$ is a nonincreasing function,

(3) $b_j(\lambda)$ has no zeros.

Then, for $1 \le i \le n$, $b_{i-1} - b_{i-1}^2 \frac{S_{i-1}}{S_i}$ is a strictly decreasing function on any interval which contains no zeros of $S_i(\lambda)$.

As before, condition (3) is automatically satisfied here, and conditions (1) and (2) are implied by the monotonicity assumptions (M1) - (M3). We shall use the following notation:

$$u_i' = (u_i - u_{i-1})/h, \quad \text{for} \quad 1 \le i \le n. \tag{5.5}$$

<u>Theorem 5.3</u> (Second Comparison Theorem). Suppose that (1.4) satisfies the monotonicity conditions (M1) - (M3). Then, for $1 \le i \le n$, $p_{i-\frac{1}{2}}(\lambda)\frac{u_i'(\lambda)}{u_i(\lambda)}$ is a strictly decreasing function on any interval which contains no zeros of $u_i(\lambda)$.

<u>Proof</u>. By Theorem 4.2, $u_i = t_i S_i$, where $t_i = \beta_0/(b_{i-1}b_{i-2}\cdots b_0)$. Therefore $\frac{u_{i-1}}{u_i} = b_{i-1}\frac{S_{i-1}}{S_i}$, and

$$p_{i-\frac{1}{2}}(\lambda)\frac{u_i'}{u_i} = \frac{b_{i-1}}{h}\left[\frac{u_i - u_{i-1}}{u_i}\right] = \frac{b_{i-1}}{h}\left[1 - \frac{u_{i-1}}{u_i}\right] = \frac{1}{h}\left[b_{i-1} - b_{i-1}^2\frac{S_{i-1}}{S_i}\right].$$

The theorem now follows from Lemma 5.1. Q.E.D.

<u>Remark</u>. If $\beta_0(\lambda) \equiv 0$, then the first row and column of $A(\lambda)$ in (3.8) must be deleted, and $u_0 = 0$. The conclusion in Lemma 5.1 is changed to: $b_i - b_i^2\frac{S_{i-1}}{S_i}$ is a strictly decreasing function. (This now agrees with the notation in [7].) In this case, $b_i - b_i^2\frac{S_{i-1}}{S_i} = hp_{i+\frac{1}{2}}\frac{u_{i+1}'}{u_{i+1}}$, so the conclusion in Theorem 5.3 remains the same, except that $2 \le i \le n$. (For $i = 1$, $p_{\frac{1}{2}}\frac{u_1'}{u_1} = \frac{b_0}{h}$, which is a nonincreasing function.)

The following lemma is implied by Lemmas 3.4 and 4.2 in Greenberg [7].

Lemma 5.2. Suppose that (1.1) satisfies assumptions (M2) and (L1). Then $\lim_{\lambda \to \Lambda_2} S_i(\lambda) = (-1)^i \infty$, for $1 \le i \le n+1$.

Theorem 5.4 (Oscillation Theorem).

(1) Suppose that (1.1) satisfies (M1) - (M4) and (L1). Then there is a number $h_0 > 0$ and an integer m, such that for mesh size $h < h_0$, and $h = 1/n$, the discrete problem has eigenvalues $\bar{\lambda}_m < \bar{\lambda}_{m+1} < \ldots < \bar{\lambda}_{n+1}$.

(2) Suppose that (1.1) satisfies (M1) - (M4), (L1) and either (L2) or (L3). Then, for any mesh size $h = 1/n$, the discrete problem has eigenvalues $\bar{\lambda}_1 < \bar{\lambda}_2 < \ldots < \bar{\lambda}_{n+1}$.

In (1) and (2), the eigenvector $u^{(k)}$, corresponding to λ_k, has exactly $k-1$ sign changes.

Proof.

(1) If $u(\lambda)$ is a solution of the discrete initial value problem (5.3), then $u(\lambda)$ has exactly $c_n(\lambda)$ sign changes. If λ is an eigenvalue of (1.1), then $u(\lambda)$ also satisfies the last equation in (3.7), and $\det[A(\lambda)] = 0$.

Let $\Lambda_1 < \lambda_0 < \Lambda_2$, and suppose that λ_0 is not an eigenvalue of (1.1) and $N(\lambda_0) = m-1$. Theorem 4.3 implies that there is a number $h_0 > 0$. such that when $h < h_0$, then $c(h,\lambda_0) = N(\lambda_0) = m-1$. We may suppose that h_0 is small enough, so that $h_0 < 1/m$. In the following, we assume that $h < h_0$.

By Lemma 5.2, $\lim_{\lambda \to \Lambda_2} S_i(h,\lambda) = (-1)^i \infty$. This implies that for $\bar{\lambda}$ near Λ_2, $c(h,\bar{\lambda}) = n+1$. Since $c(h,\lambda_0) = m-1$, Theorem 4.1 (1) implies that the discrete problem has $c(h,\bar{\lambda}) - c(h,\lambda_0) = n-m+2$ eigenvalues in the interval $[\lambda_0,\bar{\lambda})$. We shall denote them by $\bar{\lambda}_m < \bar{\lambda}_{m+1} < \ldots < \bar{\lambda}_{n+1}$, where $\lambda_0 \le \bar{\lambda}_m$ and $\bar{\lambda}_{n+1} < \bar{\lambda}$. Since $\bar{\lambda}_k$ is the only eigenvalue in $[\bar{\lambda}_k, \bar{\lambda}_{k+1})$,

$c(h,\bar\lambda_{k+1}) = c(h,\bar\lambda_k) + 1$. Also $c(h,\bar\lambda_{n+1}) = c(h,\bar\lambda) - 1 = n$. This implies that $c(h,\bar\lambda_k) = k-1$, for $m \le k \le n+1$. At any eigenvalue $\bar\lambda_k$, $S_{n+1}(h,\bar\lambda_k) = \det[A(h,\bar\lambda_k)] = 0$. Therefore $c(h,\bar\lambda_k) = c_n(h,\bar\lambda_k)$. This shows that the eigenvector $u^{(k)} = u(\bar\lambda_k)$ has exactly $k-1$ sign changes.

(2) If (1.1) also satisfies (L2) or (L3) then Lemma 4.2 implies that λ_0 can be chosen so that $c(h,\lambda_0) = 0$, for small h. In this case, $m = 1$.

Q.E.D.

We shall conclude this section with a discussion of the Sturm separation theorem. This theorem does not involve a parameter λ. It asserts that if $u(x)$ and $v(x)$ are linearly independent solutions of

$$(p(x)y')' + q(x)y = 0, \tag{5.6}$$

then the zeros of $u(x)$ and $v(x)$ are interlaced. Let C be the $(n-1)\times(n+1)$ matrix, corresponding to $A(\lambda)$ in (3.8), with the first and last rows omitted:

$$C = \begin{bmatrix} -b_0 & c_1 & -b_1 & & & & \\ & -b_1 & c_2 & -b_2 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & -b_{n-3} & c_{n-2} & -b_{n-2} & \\ & & & & -b_{n-2} & c_{n-1} & -b_{n-1} \end{bmatrix}, \tag{5.7}$$

where $c_i = b_{i-1} + b_i - a_i$. Here, the coefficients b_i, c_i are constants, with $b_i > 0$. The discrete problem corresponding to (5.6) is

$$Cy = 0, \tag{5.8}$$

where $y = (y_0, y_1, \ldots, y_n)^T$. As before, if y is a nontrivial solution of (5.8), then two consecutive terms y_i, y_{i+1} cannot both be zero. If $y_i = 0$ (where $1 \le i \le n-1$), then $y_{i-1}y_{i+1} < 0$. However, in the present situation, the possibility that $y_0 = 0$ is not excluded. We continue to use Definition 5.2 for the position of a sign change. In particular, if $y_0 = 0$, then y has a sign change at y_1.

Definition 5.3. The discrete Wronskian of $u = (u_0, u_1, \ldots, u_n)^T$ and $v = (v_0, v_1, \ldots, v_n)^T$ is $w(u,v) = (w_0, w_1, \ldots, w_{n-1})$, where $w_i = u_i v_{i+1} - v_i u_{i+1}$, for $0 \le i \le n-1$.

Lemma 5.3. Let $u = (u_0, u_1, \ldots, u_n)^T$ and $v = (v_0, v_1, \ldots, v_n)^T$ be solutions of (5.8), and let $w(u,v) = (w_0, w_1, \ldots, w_{n-1})$.

(1) u and v are linearly dependent if and only if $w(u,v) = 0$.

(2) $b_i w_i = b_0 w_0$, for $1 \le i \le n-1$.

(3) If u and v are linearly independent, then $w_i \neq 0$ for $0 \le i \le n-1$, and the w_i all have the same sign.

Proof.

(1) We may assume that neither u nor v is the zero vector. If u and v are linearly dependent, then there is a constant c such that $v = cu$. Then $w_i = u_i v_{i+1} - v_i u_{i+1} = u_i c u_{i+1} - c u_i u_{i+1} = 0$. Conversely, suppose that $w(u,v) = 0$. Then $u_i v_{i+1} - v_i u_{i+1} = 0$, for $0 \le i \le n-1$. If $v_i \neq 0$ for $0 \le i \le n$, then $\frac{u_i}{v_i} = \frac{u_{i+1}}{v_{i+1}}$. Thus $\frac{u_0}{v_0} = \frac{u_1}{v_1} = \ldots = \frac{u_n}{v_n} = c$, and $u = cv$. If $v_i = 0$, then the relation $u_i v_{i+1} - v_i u_{i+1} = 0$ implies that $u_i = 0$, since $v_{i+1} \neq 0$. (Or, if $i = n$, $u_{n-1} v_n - v_{n-1} u_n = 0$ implies that $u_n = 0$.) Let $c = \frac{u_{i+1}}{v_{i+1}}$ (or if $i = n$, $c = \frac{u_{n-1}}{v_{n-1}}$), and let $y = u - cv$. Then y is a solution of (5.8) and two consecutive terms y_i, y_{i+1} are zero. Therefore $y = 0$ and $u = cv$.

(2) The equations (5.8) for u and v are of the form

$$(5.9) \qquad \begin{cases} -b_{j-1}u_{j-1} + c_j u_j - b_j u_{j+1} = 0, \\ -b_{j-1}v_{j-1} + c_j v_j - b_j v_{j+1} = 0. \end{cases}$$

Multiply the first equation in (5.9) by v_j, and the second equation by u_j, and then subtract. This gives the equation $b_{j-1}w_{j-1} = b_j w_j$, which implies that $b_j w_j = b_{j-1}w_{j-1} = b_{j-2}w_{j-2} = \cdots = b_0 w_0$.

(3) This follows immediately from (2). Q.E.D.

Theorem 5.5 (Separation Theorem). Let $u = (u_0, u_1, \ldots, u_n)^T$ and $v = (v_0, v_1, \ldots, v_n)^T$ be linearly independent solutions of (5.8). Suppose that u has a sign change at u_i and at u_j (where $i < j$), and no sign changes between these. Then v has a sign change at some v_k, where $i \le k \le j$.

Proof. Since u cannot have a sign change at u_0, $0 < i$. The sign changes at u_i and u_j imply that $u_{i-1}u_i \le 0$, $u_i \ne 0$, $u_{j-1}u_j \le 0$ and $u_j \ne 0$. Since u has no sign changes between u_i and u_j, we may suppose that the terms $u_i, u_{i+1}, \ldots, u_{j-2}$ are all positive $u_{i-1} \le 0$, $u_{j-1} \ge 0$ and $u_j < 0$. Suppose that v has no sign change in the indicated range. Then we may suppose that $v_{i-1}, v_i, \ldots, v_{j-2}, v_{j-1}$ are all positive, and $v_j \ge 0$. Let $w(u,v) = (w_0, w_1, \ldots, w_{n-1})$. Then $w_{i-1} = u_{i-1}v_i - v_{i-1}u_i < 0$ and $w_{j-1} = u_{j-1}v_j - v_{j-1}u_j > 0$. This contradicts Lemma 5.3 (3). Q.E.D.

6. An Incorrect Algorithm.

We shall consider an alternative algorithm for calculating the n^{th} eigenvalue of (1.1). Let $u(x,\lambda)$ denote a solution of the initial value problem (1.4), and let

$$\bar{u}(x,\lambda) = \alpha_1(\lambda)u(x,\lambda) + \beta_1(\lambda)u'(x,\lambda). \tag{6.1}$$

It might seem natural to count the zeros of $\bar{u}(x,\lambda)$ (for fixed λ), rather than counting the zeros of $u(x,\lambda)$, with a correction using $\bar{u}(1,\lambda)$, as in §2. In other words, if

(6.2) $N'(\lambda)$ equals the number of zeros of $\bar{u}(x,\lambda)$ in the interval $(0,1)$,

we might conjecture the following:

Hypothetical Theorem. If (1.1) satisfies the monotonicity assumptions (M1) - (M4), and either the limit assumption (L2) or (L3), then for $\Lambda_1 < \lambda < \Lambda_2$, (1.1) has exactly $N'(\lambda)$ eigenvalues in the interval $[\Lambda_1,\lambda)$.

This would lead to an alternative shooting method for finding λ_n. However, the above Hypothetical Theorem turns out to be false. Indeed, it is already false for linear eigenvalue problems. It depends on monotonicity properties of eigenvalues, which are valid for Dirichlet boundary conditions, but not for general boundary conditions. Because of this, the alternative shooting method does not always work. We shall presently given an example where the alternative shooting method fails. This section has been included, because we believe that it is useful to point out that a numerical method can fail. It is especially important for this method, which seems to attract believers easily, and which may be used in applications.

Note that if $\bar{u}(x_0,\lambda) = 0$, then λ is an eigenvalue on the interval

$[0,x_0]$, and x_0 is a critical length (as used in the invariant imbedding method). Thus $N'(\lambda)$ is the number of critical lengths which correspond to λ (and which are less than 1).

We shall now indicate the role played by monotonicity properties of eigenvalues. We shall consider linear eigenvalue problems. In this case, (1.1) has the form

$$\text{(6.3)} \qquad \begin{cases} (p(x)u')' + (\lambda r(x)-q(x))u = 0, & \text{for } 0 \le x \le 1, \\ \alpha_0 u(0) + \beta_0 u'(0) = 0, \\ \alpha_1 u(1) + \beta_1 u'(1) = 0, \end{cases}$$

where $r(x) \ge m > 0$, and α_i, β_i are constants (for $i = 0,1$). For $0 < y \le 1$, let $\lambda_n(y)$ denote the n^{th} eigenvalue of the problem:

$$\text{(6.4)} \qquad \begin{cases} (p(x)u')' + (\lambda r(x)-q(x))u = 0, & \text{for } 0 \le x \le y, \\ \alpha_0 u(0) + \beta_0 u'(0) = 0, \\ \alpha_1 u(y) + \beta_1 u'(y) = 0. \end{cases}$$

For given n, we ask the following question:

(Q) Is $\lambda_n(y)$ a decreasing function on $(0,1]$?

If the answer to (Q) is yes, for all n, then the Hypothetical Theorem is true and the alternative algorithm is correct. To verify this, let $u(x,\lambda_0)$ be the solution of the initial value problem

$$\text{(6.5)} \qquad \begin{cases} (p(x)u')' + (\lambda_0 r(x)-q(x))u = 0, & \text{for } 0 \le x \le 1, \\ u(0) = \beta_0, \ u'(0) = -\alpha_0, \end{cases}$$

and let $\bar{u}(x,\lambda_0) = \alpha_1 u(x,\lambda_0) + \beta_1 u'(x,\lambda_0)$. If x_0 is a zero of $\bar{u}(x,\lambda_0)$ in $(0,1)$, then λ_0 is an eigenvalue on $[0,x_0]$. In other words, $\lambda_0 = \lambda_k(x_0)$, for some k. Since $\lambda_k(y)$ is a decreasing function, $\lambda_k = \lambda_k(1) < \lambda_k(x_0) = \lambda_0$. In this way, each zero of $\bar{u}(x,\lambda_0)$ corresponds to an eigenvalue less than λ_0. This shows that the Hypothetical Theorem and alternative algorithm are correct, if $\lambda_n(y)$ is a decreasing function, for all n. Unfortunately, this is not true for general boundary conditions.

If the boundary condition at the right endpoint is a Dirichlet condition: $u(y) = 0$, then the classical monotonicity theorem tells us that $\lambda_n(y)$ is a decreasing function, for all n. But Greenberg [8] has shown that if the boundary condition at the right endpoint is not a Dirichlet condition (i.e., $\beta_1 \neq 0$), then for given $n_0 \geq 1$, there exist coefficient functions $p(x)$, $q(x)$, $r(x)$ and a subinterval $[a,b] \subset (0,1]$, so that the eigenvalues $\lambda_1(y)$, $\lambda_2(y),\ldots,\lambda_{n_0}(y)$ are increasing functions in $[a,b]$. (On the other hand, for given $p(x)$, $q(x)$, $r(x)$, α_0, β_0, α_1, β_1, there exists $n_1 \geq 1$, so that for $n \geq n_1$, $\lambda_n(y)$ is a decreasing function on $(0,1]$.) THus, we cannot expect the Hypothetical Theorem and alternative algorithm to be correct for general boundary conditions. We now give a concrete example where they fail.

Example. For $0 < y \leq 1$, consider the eigenvalue problem

$$(6.6)\qquad \begin{cases} (p(x)u')' + \lambda u = 0, & \text{for } 0 \leq x \leq y, \\ u(0) + u'(0) = 0, \\ u'(y) = 0. \end{cases}$$

The energy norm is given by

$$(6.7)\qquad B(v,v) = -p(0)v(0)^2 + \int_0^y p(x)v'(x)^2dx,$$

and

$$(6.8)\qquad \lambda_1(y) = \inf_{v\in C^1[0,y]} \frac{B(v,v)}{\|v\|^2},$$

where $\|v\|^2 = \int_0^y v(x)^2\ dx$. Putting $v(x) \equiv 1$, we find that $\lambda_1(y) \le \frac{B(v,v)}{\|v\|^2} = \frac{-p(0)}{y}$. Thus

$$\lambda_1(y) < 0. \tag{6.9}$$

We now consider the two algorithms (given by Theorem 2.1 and the Hypothetical Theorem) for finding the number of eigenvalues $\lambda < 0$ (for the interval $[0,y]$). We must solve the initial value problem:

$$\begin{cases} -(p(x)u')' = 0 \\ u(0) = 1,\ u'(0) = -1. \end{cases} \tag{6.10}$$

Denote the solution by $u(x)$, and let $\bar{u}(x) = u'(x)$. We obtain: $-pu' = \text{constant} = p(0)$, so that

$$\begin{cases} \bar{u}(x) = u'(x) = \frac{-p(0)}{p(x)}, \\ u(x) = 1 - \int_0^x \frac{p(0)}{p(t)} dt. \end{cases} \tag{6.11}$$

Since $u'(x) < 0$, $u(x)$ is a decreasing function (with $u(0) = 1$). For a given y $(0 < y \le 1)$, $u'(y) < 0$ and either

(A) $u(y) \ge 0$, or (B) $u(y) < 0$.

In case (A), N_0 = [number of zeros of $u(x)$ in $(0,y)$] = 0, $\sigma = 1$, and $N = N_0 + \sigma = 1$. In case (B), $N_0 = 1$, $\sigma = 0$, and $N = N_0 + \sigma = 1$. Thus we see that the algorithm of Theorem 2.1 counts one negative eigenvalue on $[0,y]$, for all y in the interval $(0,1]$.

On the other hand, the alternative algorithm, based on the Hypothetical Theorem counts the zeros of $\bar{u}(x) = u'(x)$ in $(0,y)$. Since $\bar{u}(x) < 0$, $N' = 0$, predicting no negative eigenvalues! Here we have an example where the Hypothetical Theorem and alternative algorithm are incorrect.

Remark 1. Greenberg [8] has shown that in the above example, $\lambda_1(y)$ is an increasing function on $(0,1]$. Thus we have "reverse monotonicity" in this example!

Remark 2. An error of only 1 in the integer function $N'(\lambda)$ can have a fatal effect, when the bisection method uses $N'(\lambda)$ instead of $N(\lambda)$, as described in §2. However, the error can be much larger than 1, if some eigenvalue $\lambda_i(y)$ oscillates about a value $\lambda = \lambda_0$.

Remark 3. The eigenvalues and critical lengths can be understood geometrically by introducing polar coordinates in the phase plane (see Scott, Shampine and Wing [16]). This is usually called the Prüfer transformation. Let

$$
\text{(6.12)} \qquad \begin{cases} u = u(x,\lambda) = r(x,\lambda)\cos(x,\lambda), \\ v = p(x,\lambda)u'(x,\lambda) = r(x,\lambda)\sin\theta(x,\lambda). \end{cases}
$$

The point $U(x,\lambda) = (u(x,\lambda), p(x,\lambda)u'(x,\lambda))$ moves along a curve in the (u,v)-plane as x varies, with λ fixed, or as λ varies, with x fixed. Denoting $\theta'(x,\lambda) = \dfrac{\partial\theta(x,\lambda)}{\partial x}$ and $r'(x,\lambda) = \dfrac{\partial r(x,\lambda)}{\partial x}$ (which conforms to our notation $u'(x,\lambda) = \dfrac{\partial u(x,\lambda)}{\partial x}$), we have

$$
\text{(6.13)} \qquad \begin{cases} u' = r'\cos\theta - r\theta'\sin\theta, \\ (pu')' = r'\sin\theta + r\theta'\cos\theta. \end{cases}
$$

Setting $pu' = r\sin\theta$ and $(pu')' + qu = 0$, equations (6.13) imply

$$
\text{(6.14)} \qquad \begin{cases} pr'\cos\theta - pr\theta'\sin\theta = r\sin\theta, \\ r'\sin\theta + r\theta'\cos\theta = -qr\cos\theta. \end{cases}
$$

We can solve for θ' in (6.14), to obtain

$$
\text{(6.15)} \qquad \theta' = -[q\cos^2\theta + \frac{1}{p}\sin^2\theta].
$$

Also, (6.12) implies

(6.16) $$\tan \theta(x,\lambda) = p(x,\lambda)\,\frac{u'(x,\lambda)}{u(x,\lambda)}.$$

Equation (6.16) and Sturm's second comparison theorem show that as λ increases, with x fixed, the point $U(x,\lambda)$ moves in a clockwise direction. On the other hand, (6.15) shows that as x increases from 0 to 1, with λ fixed, $U(x,\lambda)$ may move either in the clockwise or counterclockwise direction (if $q < 0$). However, when $U(x,\lambda)$ crosses the v-axis, it moves in a clockwise direction.

We shall now consider the boundary conditions in (1.1). Let

(6.17) $$\begin{cases} \tan \eta_0(\lambda) = -p(0,\lambda)\dfrac{\alpha_0(\lambda)}{\beta_0(\lambda)}, \\ \tan \eta_1(\lambda) = -p(1,\lambda)\dfrac{\alpha_1(\lambda)}{\beta_1(\lambda)}. \end{cases}$$

Let $H_i(\lambda)$ be the line $v = [\tan \eta_i(\lambda)]u$ in the (u,v)-plane (for $i = 0,1$). The boundary condition $\alpha_0(\lambda)u(0) + \beta_0(\lambda)u'(0) = 0$ in (1) is equivalent to $\tan \theta(0,\lambda) = \tan \eta_0(\lambda)$, which means that the point $U(0,\lambda)$ lies on the line $H_0(\lambda)$. Similarly, the boundary condition $\alpha_1(\lambda)u(1) + \beta_1(\lambda)u'(1) = 0$ means that $U(1,\lambda)$ lies on $H_1(\lambda)$. Since $u(x,\lambda)$ satisfies the boundary condition at $x = 0$, we see that λ is an eigenvalue of (1.1) if and only if $U(1,\lambda)$ lies on $H_1(\lambda)$. On the other hand, x_0 is a critical length for λ if $\bar{u}(x_0,\lambda) = \alpha_1(\lambda)u(x_0,\lambda) + \beta_1(\lambda)u'(x_0,\lambda) = 0$, which means that $U(x_0,\lambda)$ lies on $H_1(\lambda)$.

Consider the trajectory of the point $U(1,\lambda)$, as λ increases from Λ_1 to λ_0. $U(1,\lambda)$ always travels in the clockwise direction, while $H_1(\lambda)$ rotates in the counterclockwise direction, because of the monotonicity assumption (M4). The number of eigenvalues less than λ_0 equals the number of times $U(1,\lambda)$ crosses $H_1(\lambda)$. A zero of $u(1,\lambda)$ corresponds to a point where $U(1,\lambda)$ crosses the v-axis. By Sturm's first comparison theorem, such zeros travel to the left in $(0,1)$, and become interior zeros of $u(x,\lambda)$. Since $U(1,\lambda)$ never reverses direction, exactly one eigenvalue occurs between any two consecutive zeros of $u(1,\lambda)$. The correction term $\sigma(\lambda)$ tells us if $U(1,\lambda)$ has crossed $H_1(\lambda)$ after the last crossing of the v-axis. In effect, a proof of the shooting theorem can be given in this framework.

Now consider the trajectory of $U(x,\lambda_0)$, as x increases from 0 to 1 for fixed $\lambda = \lambda_0$. A critical length occurs each time that $U(x,\lambda_0)$ crosses $H_1(\lambda_0)$. Thus we can count the critical lengths by counting the number of times that $\tan \theta(x,\lambda_0) = \tan \eta_1(\lambda_0)$. (See equation (5.3) and inequality (5.4) in Scott, Shampine and Wing [16].) A zero of $u(x,\lambda_0)$ corresponds to a point where $U(x,\lambda_0)$ crosses the v-axis. As indicated above, such a crossing must occur in the clockwise direction. Between two crossings of the v-axis, $U(x,\lambda_0)$ must cross the line $H_1(\lambda_0)$. Thus, there is a critical length between any two zeros of $u(x,\lambda_0)$. However, $U(x,\lambda_0)$ may reverse its direction several times between two crossings of the v-axis. Therefore, there may be many critical lengths between two zeros. On the other hand, if $\lambda_1 < \lambda_0 < \mu_1$, then $U(x,\lambda_0)$ never crosses the v-axis. In this case, it may never cross $H_1(\lambda_0)$, even though $\lambda_1 < \lambda_0$. This is what happens in the example (6.6). This explains why the number of critical lengths $N'(\lambda_0)$ may be slightly smaller than $N(\lambda_0)$, or a great deal larger than $N(\lambda_0)$.

Appendix

In this appendix, we shall prove some facts about the convergence of approximate eigenvalues. The main facts are that, as $h \longrightarrow 0$, the approximate eigenvalues converge to the true eigenvalues, and to nothing else. Theorems A.1 and A.2 make this statement precise. These facts were used in §4, especially in the proof of Lemma 4.1, and the second proof of the shooting theorem. We are including these proofs here for the sake of completeness. For a fixed value of λ, consider the linear eigenvalue problem

$$\text{(A.1)} \qquad \begin{cases} (p(x,\lambda)u')' + q(x,\lambda)u + \mu u = 0, & \text{for } 0 \le x \le 1, \\ \alpha_0(\lambda)u(0) + \beta_0(\lambda)u'(0) = 0, \\ \alpha_1(\lambda)u(1) + \beta_1(\lambda)u'(1) = 0. \end{cases}$$

Here the eigenvalue is μ. The weak form of the equation is

$$\text{(A.2)} \qquad B(\lambda;u,v) = \mu\langle u,v\rangle, \quad \text{for all} \quad v \in H^1[0,1],$$

where $B(\lambda;u,v)$ is the energy inner product (3.1), and $\langle u,v\rangle = \int_0^1 uv\,dx$. The monotonicity conditions (M1) - (M4) imply that $B(\lambda;u,u)$ is a strictly decreasing function of λ. By the variational characterization of eigenvalues, the k^{th} eigenvalue $\mu_k(\lambda)$ is a continuous, strictly decreasing function of λ. The k^{th} eigenvalue λ_k of (1.1) is the unique zero of $\mu_k(\lambda)$.

We shall distinguish between two discrete problems: The pure finite element discretization, where the integrals are not replaced by quadratures; and our discretization (3.3) and (3.7), where the integrals are replaced by quadratures. Each of these corresponds to a linear eigenvalue problem. The weak formulation of the pure finite element discretization is: Find $u \in S_h$, such that

$$B(\lambda;u,v) = \mu_h\langle u,v\rangle, \quad \text{for all} \quad v \in S_h. \tag{A.3}$$

The corresponding equation for our discretization is

$$B_h(\lambda;u,v) = \bar{\mu}_h\langle u,v\rangle_h, \quad \text{for all} \quad v \in S_h, \tag{A.4}$$

where $B_h(\lambda;u,v)$ is obtained from $B(\lambda;u,v)$ by replacing the integrals by quadratures (as indicated in §3), and $\langle u,v\rangle_h$ is the trapezoid rule quadrature for $\int_0^1 uv\,dx$. If (1.1) satisfies the monotonicity assumptions (M1)-(M4), then the eigenvalues $\mu_{k,h}(\lambda)$ of (A.3) and $\bar{\mu}_{k,h}(\lambda)$ of (A.4) are continuous, strictly decreasing functions of λ. The k^{th} eigenvalue $\bar{\lambda}_{k,h}$ of our discretization (3.7) is the unique zero of $\bar{\mu}_{k,h}(\lambda)$.

The next two lemmas estimate the errors $|\mu_k(\lambda) - \mu_{k,h}(\lambda)|$ and $|\mu_{k,h}(\lambda) - \bar{\mu}_{k,h}(\lambda)|$.

<u>Lemma A.1</u>. Let $\Lambda_1 < \lambda' < \lambda'' < \Lambda_2$. For each integer $k > 0$, there is a constant $C = C(k) > 0$, so that $|\mu_k(\lambda) - \mu_{k,h}(\lambda)| \le Ch^2$, for $\lambda' \le \lambda \le \lambda''$.

<u>Proof</u>. Let $u_k = u_k(x,\lambda)$ denote an eigenfunction of (A.2) corresponding to the eigenvalue $\mu_k(\lambda)$ (such that $\|u_k\|_{H^1[0,1]} = 1$). It is known that

$$|\mu_{k,h}(\lambda) - \mu_k(\lambda)| \le C_1 \inf_{v\in S_h} \|u_k - v\|^2_{H^1[0,1]}, \tag{A.5}$$

for $\lambda' \le \lambda \le \lambda''$, where C_1 is a constant which depends only on the coefficient functions in (1.1) (see Babuška, Osborn [3]). It is also known from finite element approximation theory (see Ciarlet [6]) that

$$\inf_{v\in S_h} \|u_k - v\|_{H^1[0,1]} \le C_2 h \|u_k\|_{H^2[0,1]}. \tag{A.6}$$

Furthermore, (A.1) implies that $u_k'' = -\frac{1}{p}[p'u_k' + (q+\mu_k)u_k]$, therefore

$\|u_k''\|_{L^2[0,1]} \le C_3$, for $\lambda' \le \lambda \le \lambda''$, and

(A.7) $$\|u_k\|_{H^2[0,1]} \le C_4, \quad \text{for} \quad \lambda' \le \lambda \le \lambda''.$$

Inequalities (A.5), (A.6), (A.7) imply

(A.8) $$|\mu_{k,h}(\lambda) - \mu_k(\lambda)| \le Ch^2, \quad \text{for} \quad \lambda' \le \lambda \le \lambda''.$$

Q.E.D.

We shall now consider the error $|\mu_{k,h}(\lambda) - \bar{\mu}_{k,h}(\lambda)|$, caused by the quadratures. We shall use the modulus of continuity $m_f(\delta)$ of a continuous function $f(x)$. Recall that $m_f(\delta) = \sup\limits_{|x-y|\le\delta} |f(x) - f(y)|$. If $f(x)$ is continuous on a compact interval, then $\lim\limits_{\delta\to 0} m_f(\delta) = 0$. For a function such as $q(x,\lambda)$, which is continuous on $[0,1]\times[\lambda',\lambda'']$,

$m_q(\delta,\lambda) = \sup\limits_{|x-y|\le\delta} |q(x,\lambda) - q(y,\lambda)|$ satisfies: $\lim\limits_{\delta\to 0} m_q(\delta,\lambda) = 0$ uniformly for $\lambda' \le \lambda \le \lambda''$. Define $m_q(\delta) = \sup\limits_{\lambda'\le\lambda\le\lambda''} m_q(\delta,\lambda)$. Then $\lim\limits_{\delta\to 0} m_q(\delta) = 0$. Since $\frac{\partial p(x,\lambda)}{\delta x}$ is continuous, there is a constant $A = A(\lambda',\lambda'')$, so that $m_p(\delta,\lambda) \le A\delta$, for $\lambda' \le \lambda \le \lambda''$. Thus $m_p(\delta) \le A\delta$.

<u>Lemma A.2</u>. Let $\Lambda_1 < \lambda' < \lambda'' < \Lambda_2$. For each integer $k > 0$, there is a function $w(\delta) = w_k(\delta)$, defined for $\delta \ge 0$, such that

(1) $\lim\limits_{\delta\to 0} w(\delta) = 0$, and

(2) $|\mu_{k,h}(\lambda) - \bar{\mu}_{k,h}(\lambda)| \le w(h)$, for $\lambda' \le \lambda \le \lambda''$.

<u>Proof</u>. We shall use the Rayleigh quotients

(A.9) $$R(\lambda,u) = \frac{B(\lambda;u,u)}{\langle u,u\rangle}, \quad \bar{R}(\lambda;u) = \frac{B_h(\lambda;u,u)}{\langle u,u\rangle_h},$$

for $\lambda' \le \lambda \le \lambda''$, $u \in S_h$.

The eigenvalues $\mu_{k,h}(\lambda)$ and $\bar{\mu}_{k,h}(\lambda)$ are determined by

(A.10)
$$\mu_{k,h}(\lambda) = \min_{\dim U=k} \max_{u\in U} R(\lambda;u),$$

(A.11)
$$\bar{\mu}_{k,h}(\lambda) = \min_{\dim U=k} \max_{u\in U} \bar{R}(\lambda;u),$$

where U is a subspace of S_h. We may assume that

(A.12)
$$\langle u,u\rangle = 1, \ \|u\|^2_{H^1(0,1]} \le M \ \text{ and } \ |R(\lambda;u)| \le M$$

in (A.10). Under these assumptions, we shall estimate $|R(\lambda;u) - \bar{R}(\lambda;u)|$. We shall use the notations

(A.13)
$$\begin{cases} D = -\dfrac{\alpha_0(\lambda)}{\beta_0(\lambda)}\, p(0,\lambda)u(0)^2 + \dfrac{\alpha_1(\lambda)}{\beta_1(\lambda)}\, p(1,\lambda)u(1)^2, \\ E = \displaystyle\int_0^1 p(x,\lambda)u'(x)^2dx, \ F = \int_0^1 q(x,\lambda)u(x)^2dx, \\ G = \displaystyle\int_0^1 u(x)^2dx. \end{cases}$$

(Under our assumptions, $G = 1$, but we shall not use this until later.) Let $\bar{E}$ denote the midpoint quadrature of E, and let $\bar{F},\bar{G}$ denote the trapezoid quadratures of F,G, respectively. Furthermore, let

(A.14)
$$e = \bar{E} - E, \quad f = \bar{F} - F, \quad g = \bar{G} - G.$$

Note that

(A.15)
$$R(\lambda;u) = \frac{D+E-F}{G}, \qquad \bar{R}(\lambda;u) = \frac{D+\bar{E}-\bar{F}}{\bar{G}}.$$

Thus $\bar{R}(\lambda;u) = \dfrac{D+\bar{E}-\bar{F}}{\bar{G}} = \dfrac{D+(E+e)-(F+f)}{G+g}$. Using the Taylor expansion,

$$\frac{1}{G+g} = \frac{1}{G}\,[1 - \frac{g}{G(1+\theta)^2}],$$

where θ lies between 0 and $\frac{g}{G}$, we obtain

$$\bar{R}(\lambda;u) = \frac{1}{G}\left[1-\frac{g}{G(1+\theta)^2}\right][D+(E+e)-(F+f)]$$

$$= \frac{D+E-F}{G} - \frac{g}{G(1+\theta)^2}\left[\frac{D+E-F}{G}\right] + \frac{1}{G}\left[1-\frac{g}{G(1+\theta)^2}\right](e-f).$$

Using (A.12), this implies

$$\text{(A.16)} \qquad |\bar{R}(\lambda;u) - R(\lambda;u)| \le \frac{|g|}{(1+\theta)^2}M + \left|1-\frac{g}{(1+\theta)^2}\right||e-f|,$$

where θ lies between 0 and g. We shall now estimate $|e|, |f|$, and $|g|$.

A function $u \in S_h$ is continuous and piecewise linear on $[0,1]$. On the interval $[x_i, x_{i+1}]$, $u(x) = u_i + u_i'(x-x_i)$, where $u_i = u(x_i)$ and $u_i' = (u_{i+1}-u_i)/h$. By definition

$E = \int_0^1 p(x,\lambda)u'(x)^2dx = \sum_{i=0}^{n-1}(u_i')^2\int_{x_i}^{x_{i+1}}p(x,\lambda)dx$, and $\bar{E} = \sum_{i=0}^{n-1}(u_i')^2p_{i+\frac{1}{2}}(\lambda)h.$

Therefore $e = \bar{E} - E = \sum_{i=0}^{n-1}(u_i')^2\int_{x_i}^{x_{i+1}}(p_{i+\frac{1}{2}}(\lambda) - p(x,\lambda))dx$, and

$|e| \le m_p(\frac{h}{2})h\sum_{i=0}^{n-1}(u_i')^2$. Furthermore, $h\sum_{i=0}^{n-1}(u_i')^2 = \int_0^1 u'(x)^2dx \le \|u\|^2_{H^1[0,1]} \le M$ (by (A.12)), and $m_p(\frac{h}{2}) \le \frac{A}{2}h$. Therefore

$$\text{(A.17)} \qquad |e| \le \frac{MA}{2}h.$$

We shall consider $|g|$ next. $G = \int_0^1 u^2dx = \sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}}[u_i+u_i'(x-x_i)]^2dx =$
$\frac{h}{3}\sum_{i=0}^{n-1}(u_i^2+u_iu_{i+1}+u_{i+1}^2) = \frac{h}{2}\sum_{i=0}^{n-1}(u_i^2+u_{i+1}^2) - \frac{h^3}{6}\sum_{i=0}^{n-1}(u_i')^2$. $\bar{G} = \frac{h}{2}\sum_{i=0}^{n-1}(u_i^2+u_{i+1}^2)$.

Therefore $g = \bar{G} - G = \frac{h^3}{6}\sum_{i=0}^{n-1}(u_i')^2 \le \frac{M}{6}h^2$. Thus we have shown

(A.18) $$0 \le g \le \frac{M}{6}h^2.$$

Now consider $|f|$. Note that there is a value $\xi_i = \xi_i(\lambda)$, such that $x_i \le \xi_i \le x_{i+1}$ and $\int_{x_i}^{x_{i+1}} q(x,\lambda)u(x)^2dx = q(\xi_i,\lambda)\int_{x_i}^{x_{i+1}} u(x)^2dx$. Denote $\bar{q}_i = q(\xi_i,\lambda)$, while $q_i = q(x_i,\lambda)$. $F = \int_0^1 q(x,\lambda)u(x)^2dx = \sum_{i=0}^{n-1}\bar{q}_i\int_{x_i}^{x_{i+1}} u(x)^2dx$

$$= \sum_{i=0}^{n-1}\bar{q}_i \frac{h}{3}(u_i^2+u_iu_{i+1}+u_{i+1}^2) = \frac{h}{2}\sum_{i=0}^{n-1}\bar{q}_i(u_i^2+u_{i+1}^2) - \frac{h^3}{6}\sum_{i=0}^{n-1}\bar{q}_i(u_i')^2.$$

$$\bar{F} = \frac{h}{2}\sum_{i=0}^{n-1}(q_iu_i^2+q_{i+1}u_{i+1}^2).$$

$$f = \bar{F} - F = \frac{h}{2}\sum_{i=0}^{n-1}\left\{(q_i-\bar{q}_i)u_i^2 + (q_{i+1}-\bar{q}_i)u_{i+1}^2\right\} + \frac{h^3}{6}\sum_{i=0}^{n-1}\bar{q}_i(u_i')^2.$$ Let

$Q = \max\limits_{\substack{0\le x\le 1\\ \lambda'\le\lambda\le\lambda''}} |q(x,\lambda)|$. Then $|f| \le m_q(h)\cdot\frac{h}{2}\sum_{i=0}^{n-1}(u_i^2+u_{i+1}^2) + \frac{h^2}{6}Q\int_0^1 u'(x)^2dx \le$

$$m_q(h)h\sum_{i=0}^{n-1}(u_i^2+u_iu_{i+1}u_{i+1}^2) + \frac{MQ}{6}h^2 = m_q(h)3\int_0^1 u(x)^2dx + \frac{MQ}{6}h^2 = 3m_q(h) + \frac{MQ}{6}h^2.$$

We have shown

(A.19) $$|f| \le 3m_q(h) + \frac{MQ}{6}h^2.$$

Inequalities (A.16) - (A.19) imply

(A.20) $$|\bar{R}(\lambda;u) - R(\lambda;u)| \le w(h), \quad \text{for } \lambda' \le \lambda \le \lambda'',$$

where

(A.21) $$w(h) = \frac{M^2}{6}h^2 + (1 + \frac{M}{6}h^2)(\frac{MA}{2}h + \frac{MQ}{6}h^2 + 3m_q(h)).$$

Equations (A.10) and (A.11) now imply

$$|\bar{\mu}_{k,h}(\lambda) - \mu_{k,h}(\lambda)| \le w(h), \quad \text{for } \lambda' \le \lambda \le \lambda''.$$

Q.E.D

Theorem A.1. Suppose that (1.1) satisfies the monotonicity assumptions (M1) - (M4). Then for each eigenvalue λ_k of (1.1), there is an $\varepsilon_0 > 0$, such that for $0 < \varepsilon < \varepsilon_0$, and for small h, the discrete problem (A.4) has a unique eigenvalue $\bar{\lambda}_{k,h}$ in the interval $(\lambda_k-\varepsilon, \lambda_k+\varepsilon)$.

Proof. Let $\Lambda_1 < \lambda' < \lambda_k < \lambda'' < \Lambda_2$. Lemmas A.1 and A.2 imply that

(A.22) $$|\bar{\mu}_{k,h}(\lambda) - \mu_k(\lambda)| \le z_k(h), \quad \text{for} \quad \lambda' \le \lambda \le \lambda'',$$

where $z_k(h) = w_k(h) + C(k)h^2$. Let $z(h) = \max(z_{k-1}(h), z_k(h), z_{k+1}(h))$. Then for $j = k-1, k, k+1$,

(A.23) $$|\bar{\mu}_{j,h}(\lambda) - \mu_j(\lambda)| \le z(h), \quad \text{for} \quad \lambda' \le \lambda \le \lambda''.$$

Note that $\lim_{h\to 0} z(h) = 0$.

Now since λ_k is an eigenvalue of (1.1), $\mu_k(\lambda_k) = 0$. Since $\mu_k(\lambda)$ is a simple eigenvalue, $\mu_{k-1}(\lambda_k) < 0 < \mu_{k+1}(\lambda_k)$. Let $m = \min(|\mu_{k-1}(\lambda_k)|, \mu_{k+1}(\lambda_k))$, and choose $\varepsilon_0 > 0$ small enough so that $|\mu_{k-1}(\lambda)| \ge m/2$ and $\mu_{k+1}(\lambda) \ge m/2$ for $\lambda_k - \varepsilon_0 \le \lambda \le \lambda_k + \varepsilon_0$. Let $0 < \varepsilon < \varepsilon_0$. The monotonicity assumptions (M1) - (M4) imply that $\mu_k(\lambda)$ is strictly decreasing. Therefore $\mu_k(\lambda_k-\varepsilon) > 0 > \mu_k(\lambda_k+\varepsilon)$. Now choose h_0 small enough so that

(A.24) $$z(h) \le m/4, \quad \text{for} \quad 0 < h < h_0,$$

(A.25) $$\mu_k(\lambda_k-\varepsilon) - z(h) > 0, \quad \text{for} \quad 0 < h < h_0,$$

and

(A.26) $$\mu_k(\lambda_k+\varepsilon) + z(h) < 0, \quad \text{for} \quad 0 < h < h_0.$$

The inequalities (A.23) - (A.26) imply (for $0 < h < h_0$)

(A.27) $$\bar{\mu}_{k,h}(\lambda_k-\varepsilon) > 0, \ \bar{\mu}_{k,h}(\lambda_k+\varepsilon) < 0,$$

(A.28) $$\bar{\mu}_{k-1,h}(\lambda) < 0, \quad \text{for} \quad \lambda_k-\varepsilon \le \lambda \le \lambda_k+\varepsilon,$$

and

(A.29) $$\bar{\mu}_{k+1,h}(\lambda) > 0, \quad \text{for} \quad \lambda_k-\varepsilon \le \lambda \le \lambda_k+\varepsilon.$$

This implies that $\bar{\mu}_{k,h}(\lambda)$ has a zero $\bar{\lambda}_{k,h}$ in $(\lambda_k-\varepsilon,\lambda_k+\varepsilon)$, and for $i \ne k$, $\bar{\mu}_{i,h}(\lambda)$ has no zero in $(\lambda_k-\varepsilon,\lambda_k+\varepsilon)$. Q.E.D.

Remark. If (1.1) satisfies the monotonicity assumptions (M1) - (M4), then the pure finite element eigenvalue $\mu_{k,h}(\lambda)$ is a strictly decreasing function. Let $\lambda_{k,h}$ be the unique zero of $\mu_{k,h}(\lambda)$. The minimax principle implies that $\mu_k(\lambda) \le \mu_{k,h}(\lambda)$, and therefore $\lambda_k \le \lambda_{k,h}$. These facts may not be true for $\bar{\mu}_{k,h}(\lambda)$ and $\bar{\lambda}_{k,h}$. If $\lambda_k \le \bar{\lambda}_{k,h}$ is always true, then Theorem 4.3 is also valid in the case that λ is an eigenvalue of (1.1).

Theorem A.2. Let $\Lambda_1 < \lambda_0 < \Lambda_2$, and suppose that λ_0 is not an eigenvalue of (1.1). Then there is an $\varepsilon > 0$, such that for small h, the discrete problem has no eigenvalue in the interval $(\lambda_0-\varepsilon,\lambda_0+\varepsilon)$.

Proof. If the statement is false, then there is a sequence $h_n \to 0$ and eigenvalues $\bar{\lambda}_{k_n,h_n}$ of the discrete problem, so that $\bar{\lambda}_{k_n,h_n} \to \lambda_0$. Let $u_n(x)$ be an eigenfunction corresponding to $\bar{\lambda}_{k_n,h_n}$ (with $\|u_n\|_{H^1[0,1]} = 1$). Thus $B_{h_n}(\bar{\lambda}_{k_n,h_n};u_n,v) = 0$, for all $v \in S_{h_n}$. Since the u_n are bounded in $H^1[0,1]$, there is a subsequence (which we again denote u_n) which has a weak limit u_0 in $H^1[0,1]$. Since $h_n \to 0$ and $\bar{\lambda}_{k_n,h_n} \to \lambda_0$, it follows that $B(\lambda_0;u_0,v) = 0$ for $v \in \bigcup S_{h_n}$. Since $\bigcup S_{h_n}$ is dense in $H^1[0,1]$, this implies that $B(\lambda_0;u_0,v) = 0$ for all $v \in H^1[0,1]$. But this means that λ_0 is an eigenvalue of (1.1). Q.E.D.

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